Ω-Krull Rings, I.

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July 1981

81-26

0. Introduction

In the search for a class of noncommutative rings with an arithmetical ideal theory, generalizing the classical theory of Dedekind domains, several possibilities arise; e.g. HNP-rings, Asano orders and Dedekind prime rings. An even richer gamma of possible definitions is available in case one aims to produce a noncommutative counterpart to the theory of Krull domains. A natural way to define these rings is by imposing local conditions at the minimal nonzero prime ideals. The first ramification of the theory is created by the fact that in the noncommutative case one may consider independent local conditions on the localized rings but also on the type of localization used in the construction of that local ring. Recently, some types of noncommutative Krull rings have been studied by M. Chamarie [2], R. Fossum [3], and in particular H. Marubayashi [8,9,10,11]. All these rings are supposed to be (maximal) orders in simple Artinian rings, e.g. Marubayashi-Krull rings are Goldie prime rings such that the Lambek-Michler localizations at prime ideals of height one are Asano orders.

On the other hand, F. Van Oystaeyen constructed a class of (not

(*) Both the second and third named author are supported by an N.F.W.O.-grant.
necessarily Goldie-) prime rings having properties analogous to those of Asano-orders, namely $\Omega$-rings, cfr [13,19]. In this note we aim to generalize $\Omega$-rings in about the same manner Marubayashi Krull rings generalize Asano-orders, this time using symmetric localization.

In section 2 we prove that the divisor classes form an abelian group which is a direct product of infinite cyclic subgroups. One of the main motivations for studying $\Omega$-rings is the fact that they fit nicely in the theory of primes [13,15], which is the most manageable generalization of valuation theory to the noncommutative case known to the authors. In section 3 we indicate that there is an equally sufficient valuation-like theory attached to $\Omega$-Krull rings, which will be developed in part II of this paper.

In the last two sections, we present some examples of $\Omega$-Krull rings. A sufficient condition is given to assure that $R[T]$ remains $\Omega$-Krull if $R$ is so.
1. $\sigma$-Krull rings

Throughout this note, all rings will be associative and have a unit element, modules will be unitary. Ideal will always mean two-sided ideal. $R$-mod (resp. mod-$R$) stands for the category of all left (resp. right) $R$-modules.

An endofunctor $\sigma$ in $R$-mod is said to be a kernel functor if it is a left exact subfunctor of the identity in $R$-mod, $\sigma$ is said to be idempotent if $\sigma(M/\sigma(M)) = 0$ for any $M \in R$-mod. To a kernel functor $\sigma$ the filter of left ideals of $R$, $\mathcal{L}(\sigma) = \{L \text{ left ideal of } R : \sigma(R/L) = R/L\}$ is associated and to a filter $\mathcal{L}$ satisfying:

(K1) : If $I, J \in \mathcal{L}$, then $I \cap J \in \mathcal{L}$ ;

(K2) : If $I \in \mathcal{L}$ and $J$ is a left ideal of $R$ such that $I \subseteq J$, then $J \in \mathcal{L}$ ;

(K3) : If $I \in \mathcal{L}$ and $x \in R$, then $(I:x) = \{r \in R : rx \in I\} \in \mathcal{L}$, one associates the kernel functor $\sigma_\mathcal{L}(M) = \{m \in M : \exists I \in \mathcal{L} : Im = 0\}$. Recall from [5,14,17] that this defines a one-to-one correspondence. $\sigma_\mathcal{L}$ will be idempotent if and only if $\mathcal{L}$ satisfies also:

(K4) : If $I \in \mathcal{L}$ and $J$ is a left ideal of $R$ such that $(J:x) \in \mathcal{L}$ for every $x \in I$, then $J \in \mathcal{L}$.

A kernel functor $\sigma$ is called bilateral if its associated filter $\mathcal{L}(\sigma)$ has a cofinal set consisting of ideals, $\sigma$ is said to be symmetric if it is both idempotent and bilateral.

It is well known that one can associate to any idempotent kernel functor $\sigma$ a left exact localization functor $Q_\sigma(\cdot)$ in $R$-mod. $Q_\sigma(R)$ is a ring containing $R/\sigma(R)$ as a subring and there is a canonical ringhomomorphism $j_\sigma : R \to Q_\sigma(R)$. In particular, if $R$ is a prime ring, $\sigma(R) = 0$, $j_\sigma$ is the canonical embedding and $R$ may be viewed as a subring of $Q_\sigma(R)$. An idempotent kernel functor $\sigma$ is said to have property $T$ if it satisfies one of the following equivalent condi-
tions: (T1): \( Q_\sigma(\cdot) \) is right exact and commutes with direct sums;

(T2): For every \( I \in \mathcal{I}(\sigma) \) we have: \( Q_\sigma(R) j_\sigma(I) = Q_\sigma(R) \);

(T3): For every \( M \in R\text{-mod} : Q_\sigma(M) \cong Q_\sigma(R) \otimes_R M \);

(T4): \( j_\sigma : R \to Q_\sigma(R) \) is a flat epimorphism of rings.

\( \sigma \) will be called geometrical if it has property T and satisfies:

(G): For any ideal \( I \) of \( R \), \( Q_\sigma(R) j_\sigma(I) \) is an ideal of \( Q_\sigma(R) \). E.g. if \( \sigma \) is a central idempotent kernel functor, i.e. \( \mathcal{I}(\sigma) \) has a cofinal set consisting of centrally generated ideals, \( \sigma \) is geometrical.

Likewise, one can define all these concepts in mod-\( R \). If \( \mathcal{I}^2(\sigma) \) is a set of ideals of \( R \) which is multiplicatively closed and if \( \mathcal{I}_L^\sigma(\cdot) \) (resp. \( \mathcal{I}_R^\sigma(\cdot) \)) (i.e. the filter of left (resp. right) \( R \)-ideals generated by \( \mathcal{I}^2(\sigma) \)) is idempotent, we will denote by \( Q_\sigma^{L}(\cdot) \) (resp. \( Q_\sigma^{R}(\cdot) \)) the localization functor in \( R\text{-mod} \) (resp. \( \text{mod-R} \)) associated with \( \mathcal{I}_L^\sigma(\cdot) \) (resp. \( \mathcal{I}_R^\sigma(\cdot) \)). E.g. if \( P \) is a prime ideal of \( R \), \( \mathcal{I}^2(R-P) \) will be the multiplicatively closed set of ideals \( I \) of \( R \) not contained in \( P \), \( Q_\sigma^{L}(\cdot) \) and \( Q_\sigma^{R}(\cdot) \) will be the associated localization functors.

Throughout, \( R \) will be a prime ring. The first problem encountered is to find a symmetric analogue of the Goldie theorems, i.e. to give necessary and sufficient conditions such that \( R \) may be embedded in a symmetric localization \( Q_{\text{sym}}(R) \) which is a simple ring (eventually satisfying additional chain conditions) such that any localization of \( R \) at a symmetric kernel functor can be viewed as a subring of \( Q_{\text{sym}}(R) \). Clearly, a sufficient condition is that \( \sigma_{R-0} \) is an idempotent kernel functor having property T. However, to find a necessary and sufficient intrinsic characterization in terms of elements and ideals of \( R \) might prove rather difficult.

In order to bypass this problem as well as to exclude oddities as the ones encountered in [6] and [7], arising from the fact that certain prime ideals may have trivial intersection with the center \( Z(R) \), we will limit ourselves
to prime rings $R$ satisfying Formanek's condition:

(F) : For every ideal $I$ of $R$, $I \cap Z(R) \neq 0$.

**Lemma 1.1.** If $R$ is a prime ring satisfying (F), then:

1. $\sigma_{R-0}$ is an idempotent kernel functor,
2. $Q_{R-0}^\sigma(R) \cong Q_{R-0}^\tau(R) \cong \{c^{-1}r = rc^{-1} \mid r \in R, 0 \neq c \in Z(R)\} = Q_{sym}(R)$ are simple rings.

**proof**

1. Suppose $I \in \mathcal{L}^2(R-0)$, $J$ a left ideal of $R$ such that $\sigma(I/J) = I/J$. It will be sufficient to prove that $J \in \mathcal{L}^2(R-0)$. Take $0 \neq c \in I \cap Z(R)$, then $I' = Rc \in \mathcal{L}^2(R-0)$ and $\sigma(I'/I' \cap J) = I'/I' \cap J$. There exists an ideal $I'' \in \mathcal{L}^2(R-0)$ such that $I''c = cI'' \subset I' \cap J$ whence : $I'I'' \subset I' \cap J \subset J$ and thus $J \in \mathcal{L}^2(R-0)$ because $I'I'' \in \mathcal{L}^2(R-0)$.

2. In view of (1) it is easy to check that $\sigma_{R-0}^\sigma$ and $\sigma_{R-0}^\tau$ have property T, using results of [19]. Therefore, $Q_{R-0}^\sigma(R)$ and $Q_{R-0}^\tau(R)$ are simple rings. Verification of the fact that they are equal to $\{c^{-1}r = rc^{-1} \mid r \in R, 0 \neq c \in Z(R)\}$ is straightforward.

A prime ring satisfying (F) is said to be an $\Omega$-ring if every ideal is a product of maximal ideals. We call a ring quasi-local if it has a unique maximal ideal.

**Definition 1.2.** A prime ring satisfying (F) is said to be an $\Omega$-Krull ring if the following conditions hold:

1. There exist multiplicatively closed sets of ideals $\mathcal{L}^2(\sigma_i)$ ($i \in \Lambda$) such that:
   
   $R_i = Q_{R-0}^\sigma(R) = \{q \in Q_{sym}(R) \mid \exists I \in \mathcal{L}^2(\sigma_i), Iq \subset R\}$
   
   $\quad = Q_{R-0}^\tau(R) = \{q \in Q_{sym}(R) \mid \exists I \in \mathcal{L}^\tau(\sigma_i), qI \subset R\}$

2. $\forall i \in \Lambda : R_i$ is a quasi-local $\Omega$-ring;

3. $R = \bigcap_{i \in \Lambda} R_i$. 
(4) : For every \( r \in R \) there are only finitely many \( i \in \Lambda \) such that 
\( r = \text{RrR} \notin \mathcal{L}^2(\sigma_i) \);

(5) : \( \forall i \in \Lambda ; \forall I \in \mathcal{L}^2(\sigma_i) : R_i I = \text{IR}_i = R_i \).

Remark : We have used the notation \( R_i = Q_{\sigma_i}^e(R) = Q_{\sigma_i}^r(R) \). However, there is 
no a priori reason why these \( R_i \) should be localizations. The following lemma 
proves they are :

**Lemma 1.3.** : \( \forall i \in \Lambda : \sigma_i \) is an idempotent kernel functor.

**Proof**

(We write \( \sigma \) for \( \sigma_i \)). Suppose \( I \in \mathcal{L}^2(\sigma) \), \( J \) a left \( R \)-ideal and \( \sigma(\text{I}/\text{J}) = \text{I}/\text{J} \).

Because \( Q_{\sigma_i}^e(R)I = Q_{\sigma_i}^e(R) \) we can write :

\[
1 = \alpha_1 \beta_1 + \ldots + \alpha_n \beta_n \]

where \( \alpha_1 \in Q_{\sigma_i}^e(R) \) and \( \beta_i \in I \). Take, \( I' = R\beta_1 + \ldots + R\beta_n \) 
then \( Q_{\sigma_i}^e(R)I' = Q_{\sigma_i}^e(R) \). Therefore \( I' \in \mathcal{L}^2(\sigma) \) for there exists an ideal \( K \in \mathcal{L}^2(\sigma) \) 
such that \( K\alpha_i \subset R, 1 \leq i \leq n \). Therefore \( K.I \subset R\beta_1 + \ldots + R\beta_n = I' \). Because 
\( I' \subset I \) we have \( \sigma(\text{I'}/\text{J} \cap \text{I'}) = \text{I'}/\text{J} \cap \text{I'} \). So, there exists an ideal \( L \in \mathcal{L}^2(\sigma) \) 
such that \( L\beta_i \subset J \cap I' \), \( 1 \leq i \leq n \). Finally, we obtain : \( L.I' \subset J \cap I' \subset J \) 
and \( L.I' \in \mathcal{L}^2(\sigma) \) because \( \mathcal{L}^2(\sigma) \) is closed under taking products, therefore 
\( J \in \mathcal{L}^2(\sigma) \).

We immediately obtain that \( R_i = Q_{\sigma_i}^e(R) \) is indeed the localization of \( R \) with 
respect to the symmetric kernel functor \( \sigma_i^e \) having property (T). (5) may be 
strengthened to :

(5') : \( \forall i \in \Lambda : \sigma_i^e \) and \( \sigma_i^r \) are geometrical, or to :

(5'') : \( \forall i \in \Lambda : \sigma_i^e \) and \( \sigma_i^r \) are central kernel functors.

Remark : The \( R_i \) are prime rings satisfying the condition of Formanek, so they 
have a symmetric ring of quotients \( Q_{\text{sym}}(R_i) \). If \( R \) satisfies (5') it can be 
seen (using results of [16]) that \( Q_{\text{sym}}(R) = Q_{\text{sym}}(R_i) \).
Each $R_i$ is a quasi-local $\Omega$-ring. In particular, $R_i$ has a unique maximal ideal $P'_i$, we will denote $P_i = R \cap P'_i$.

**Proposition 1.4.**

(1) : $P'_i = P_i R_i = R_i P_i$.

(2) : $P_i$ is a prime ideal of $R$.

**proof**

(1) : It is clear that $R_i P_i \subset P'_i$. Conversely, take $a \in P'_i$ then we can find an ideal $I \in \mathcal{L}^2(\sigma_i)$ such that $I a \subset R \cap P'_i = P_i$. Hence $a \in R_i a = R_i I a \subset R_i P_i$.

Similarly, $P'_i = P_i R_i$.

(2) : Suppose $AB \subset P_i$ where $A$ and $B$ are ideals of $R$ and $B \not\subset P_i$. Then $R_i BR_i \not\subset P'_i$ whence $R_i BR_i = R_i$. Write $1 = \sum x_j b_j y_j$ where $x_j, y_j \in R_i$ and $b_j \in B$. There exists an ideal $I \in \mathcal{L}^2(\sigma_i)$ such that $Ix_j \subset R$ and $y_j I \subset R$ for each $j$. Therefore, $I^2 = I.I \subset RBR = B$. Finally, we obtain $A \subset AR_i = A \cap I^2 R_i \subset ABR_i \subset P_i R_i = P'_i$ and thus : $A \subset P'_i \cap R = P_i$.

**Remark**:

1. In case $R$ satisfies $(S')$, Prop. 1.4. is nothing but a reformulation of the following result due to F. Van Oystaeyen [16] : For a geometrical kernel functor $\kappa$ there is a one-to-one correspondence between $\text{Spec} \, Q_\kappa(R)$ and $G(\kappa)$ the set of prime ideals of $R$ maximal with respect to not belonging to $\mathcal{L}(\kappa)$.

2. It is straightforward to check that $\mathcal{L}^2(\sigma_i) \subset \mathcal{L}^2(R - P_i)$. Conversely, if $J \in \mathcal{L}^2(R - P_i)$ then $J \not\subset P_i$. Take $x \in J - P_i$, then $: R_i x R_i = R_i$. Therefore we can find elements $a_k b_k$ in $R_i$ such that $\sum a_k x b_k = 1$. There exists an ideal $I \in \mathcal{L}^2(\sigma_i)$ such that $I a_k \subset R$ and $b_k I \subset R$ for all $k$. Finally, $I^2 = I.I \subset R \times R \subset J$ whence $J \in \mathcal{L}^2(\sigma_i)$.

Now consider the following conditions :
\( (6) : \forall i \neq j \in \Lambda : P_i \not\subset P_j \text{ and } P_j \not\subset P_i; \)

\( (6') : \forall i \neq j \in \Lambda : P_i R_j = P_j P_i = R_j. \)

Using remark 2 above it is easy to see that these conditions are equivalent, for \( P_i \not\subset P_j \iff P_i \in L^2(R-P_j) = L^2(\sigma_j) \iff P_i R_j = R_j P_i = R_j. \)

2. Fractional ideals and divisor classes

For every \( i \in \Lambda \), we define a fractional \( R_i \)-ideal \( I_i \) to be a nonzero left and right \( R_i \)-submodule of \( Q_{\text{sym}}(R) \) such that there exists an element \( c_i \in Z(R_i) : c_i I_i \subset R_i. \)

**Proposition 2.1.**: The fractional ideals of \( R_i \) form an abelian group under multiplication, for every \( i \in \Lambda. \)

**proof**

Take \( 0 \neq c \in P_i' \cap Z(R_i). \) Then \( R_i c \cdot R_i c^{-1} = R_i = R_i c^{-1} \cdot R_i c \) (remark that \( c \) is invertible in \( Q_{\text{sym}}(R) \) because \( Z(R_i) \subset Z(Q_{\text{sym}}(R)) \) which is a field because \( Q_{\text{sym}}(R) \) is simple). \( R_i c \) is an ideal of \( R_i \), hence, \( R_i c = (P_i')^n \) for some \( n \in \mathbb{N}. \) Therefore, \( P_i' \cdot ((P_i')^{n-1} \cdot R_i c^{-1}) = R_i = (R_i c^{-1} \cdot (P_i')^{n-1}). P_i' \) and thus \( P_i'^{-1} \) exists. This implies that every ideal of \( R_i \) is invertible. That the fractional ideals of \( R_i \) form an abelian group is now easily checked.

We define a fractional \( R \)-ideal \( I \) to be a nonzero left and right \( R \)-submodule of \( Q_{\text{sym}}(R) \) such that there exists an element \( c \in Z(R) : c I \subset R. \)

**Lemma 2.2.**: When \( I \) is a fractional \( R \)-ideal, \( R_i I R_i \neq R_i \) for only finitely many \( i \in \Lambda. \)

**proof**

Because \( Q_{\text{sym}}(R) \) is an essential extension of \( R, I \cap R \neq 0. \) Take \( 0 \neq c \in I \cap R, \)
then \((c) = RcR \subseteq I\) and \(R_iR_i' \subseteq R_iIR_i\). From (4) and (5) we get that
\(R_i \subseteq R_iIR_i\) for almost all \(i \in \Lambda\). On the other hand, there is an element
\(d \in Z(R)\) such that \(dR_iIR_i \subseteq R_i\) for each \(i \in \Lambda\). Now, \(R_i(R_iR) = R_i\) for
almost all \(i \in \Lambda\). Hence \(R_iIR_i \subseteq R_i\) for all but finitely many \(i \in \Lambda\).

**Lemma 2.3.** Suppose that \(R\) satisfies (6) and let \(I_i'\) be a fractional \(R_i\)-ideal
for each \(i \in \Lambda\) such that for almost all \(i \in \Lambda\): \(I_i' = R_i\). Then \(I = \bigcap I_i'\) is a
fractional \(R\)-ideal and \(R_iI = IR_i = I_i'\).

**Proof.**

Put \(I_i = I_i' \cap R\). Suppose first that \(I_i' \subseteq R_i\) for every \(i \in \Lambda\). Then \(I \subseteq R\)
and \(I = I_1 \cap \ldots \cap I_k\) with \(I_j \subseteq R_j\), \(j \in \{1, \ldots, k\}\). We have that each
\(I_j = (P_j)^{n_j}(n_j > 0)\) whence \((P_j)^{n_j} \subseteq I_j = (P_j)^{n_j} \cap R\). When \(i \neq j\), then
\(R_i = (P_j)^{n_j} R_i \subseteq I_j R_i \subseteq R_i\) because of (6'). When \(i = j\), \(R_j I_j = I_j'\). Because
\(R_i\) is a flat \(R\)-module, we get that \(R_i I = R_i(I_1 \cap \ldots \cap I_k) = R_i I_1 \cap \ldots \cap R_i I_k = R_i \cap \ldots \cap I_i' \cap \ldots \cap R_i = I_i' = IR_i\). In the general case, there exist elements
\(c_i \in Z(R_i)\) for each \(1 \leq i \leq k\) such that \(c_i I_i' \subseteq R_i\). Furthermore, all \(c_i\) may
be chosen in \(Z(R)\) because \(Z(R_i) \subseteq Z(Q_{sym}(R))\) and \(Z(Q_{sym}(R))\) is the field of
fractions of \(Z(R)\). Put \(c = c_1 \ldots c_k \in Z(R)\) then \(c I = \bigcap c I_i'\) and hence
\(c R_i = c I_i'\) whence \(I_i' = I_i'\).

Let \(A\) be a fractional \(R\)-ideal. Consider the twosided \(R\)-module
\(A = \bigcap _{i \in \Lambda} R_i A R_i\). Clearly, \(A\) is a fractional \(R\)-ideal and we can write each
\(R_i A R_i = (P_i')^{n_i}(n_i \in Z)\), whence \(A = \bigcap _{i \in \Lambda} (P_i')^{n_i}\) and almost all \(n_i\) are equal
to 0.

**Lemma 2.4.** (1) \(A \subseteq A_d\); (2) If \(A \subseteq B\) then \(A_d \subseteq B_d\); (3) \(A_{dd} = A_d\).

**Proof.**

(1) and (2) are obvious.

(3) \(A_d = \bigcap (P_i')^{n_i}\). Since each \((P_i')^{n_i}\) is a fractional \(R_i\)-ideal, \(R_i A R_i = (P_i')^{n_i}\)
by lemma 2.3. Hence $A_{dd} = A_d$.

**Remark:** It is easy to check that $A_d$ is precisely the $\wedge_{i \in \Lambda} \sigma_i$-closure of $A$ in $Q_{\text{sym}}(R)$, i.e.:

$$A_d = \{ q \in Q_{\text{sym}}(R) | \forall i \in \Lambda, \exists I \in L^2(\sigma_i) : Iq \subset A \text{ and } qI \subset A \}.$$ 

If $A = A_d$, then $A$ is said to be a **divisorial ideal**. We define an equivalence relation on the set of fractional $R$-ideals by saying that $A \sim B$ if and only if $A_d = B_d$. Denote by $\bar{A}$ the equivalence class determined by $A$. The set $D(R)$ of all equivalence classes forms a semigroup under $\times$ defined by $\bar{A} \times \bar{B} = (A_dB_d)$.

The unit element of $D(R)$ is of course $\bar{R}$.

**Theorem 2.5.** If $R$ is an $\Omega$-Krull ring satisfying (6), then $D(R)$ is a direct product of infinite cyclic subgroups generated by $\{R_i\}_{i \in \Lambda}$.

**Proof**

1. Suppose $A$ is a divisorial ideal, say $A = \bigcap P_i^{n_i}$ where $n_i \in \mathbb{Z}$ and $n_i = 0$ for almost all $i \in \Lambda$. Put $B = \bigcap P_i^{-n_i}$. From lemma 2.3 we deduce that $B$ is a divisorial $R$-ideal. Then, $R_i(R_iAR_i)(R_iBR_i)R_i = (P_i^{n_i}P_i^{-n_i}) = R_i$. Hence, $\bar{A} \times \bar{B} = \bar{R}$ and $D(R)$ is a group under $\times$.

2. Suppose $A = \bigcap P_i^{n_i}$ and $B = \bigcap P_i^{m_i}$ are both divisorial $R$-ideals, then $R_i(R_iAR_i)(R_iBR_i)R_i = P_i^{n_i+m_i} = R_i(R_iBR_i)(R_iAR_i)R_i$ whence $D(R)$ is abelian.

3. Write $A = P_1^{n_1} \cap \ldots \cap P_k^{n_k} \cap (\cap R_j)$. It is easy to check that $\bar{A} = (\bar{P}_1)^{n_1} \times \ldots \times (\bar{P}_k)^{n_k}$, whence $D(R)$ is generated by $\{\bar{P}_i\}_{i \in \Lambda}$. Finally, suppose $(\bar{P}_1)^{n_1} \times \ldots \times (\bar{P}_n)^{n_k} = \bar{R}$, then $P_1^{n_1} \cap \ldots \cap P_k^{n_k} \cap (\cap R_j) = R$ whence $R_iR = R_i = (P_i^{n_i})$ yielding that every $n_i = 0$.

**Theorem 2.6.** The center of an $\Omega$-Krull ring is a Krull domain.

**Proof**

Because $R$ is a prime ring, $Z(R)$ is clearly a domain. Because the $R_i$ are
localizations of $R_i$, it is readily checked that $Z(R) = \bigcap_{i \in \Lambda} Z(R_i)$. We will prove that each $Z(R_i)$ is a discrete valuation ring. We know that $K = Z(Q_{sym}(R))$ is the field of fractions of $Z(R_i)$. Define a function $v$ on $K^\times$ in the following way:

$$v : K^\times \to \mathbb{Z} : a^{-1}b \to n-m \text{ where } R_i a = (P_i')^m; R_i b = (P_i')^n.$$ Suppose $a^{-1}b = c^{-1}d$ where $a, b, c$ and $d$ are elements of $Z(R_i)$. We can write $R_i c = (P_i')^k$ and $R_i d = (P_i')^\ell$. Because the elements of $Z(R_i)$ are invertible in $K$, $R_i a R_i a^{-1} = R_i$ whence $R_i a^{-1} = (P_i')^{-m}$, similarly $R_i c^{-1} = (P_i')^{-k}$. Therefore $R_i a^{-1} R_i b = R_i c^{-1} R_i d$ whence $(P_i')^{n-m} = (P_i')^{\ell-k}$. Thus $n-m = \ell-k$ and $v$ is well defined. It is now easy to check that $v$ is a $Z$-valued valuation. $v(a^{-1}b) > 0$ if and only if $R_i a^{-1}b = (P_i')^{n-m} \subseteq R_i$, whence $a^{-1}b \in R_i$. We conclude that $\{x \in K^\times | v(x) > 0\} = Z(R_i)$ and therefore $Z(R_i)$ is a discrete valuation ring.

Finally, take $c \in Z(R)$. Then $R_i (c) = R_i$ for almost all $i \in \Lambda$. But $(c) = R_c$ and $R_i (c) = R_i c = R_i$ for almost all $i \in \Lambda$. Hence $c$ is a unit in all but finitely many $i$ whence $Z(R)$ is a Krull domain.

Let $A$ be a ring, $X = \text{Spec } A$ its prime spectrum equipped with the Zariski topology. A Zariski open set is equal to some $X(I) = \{P \in X | I \not\subseteq P\}$ where $I$ is an ideal of $A$. A ring homomorphism $f : A \to B$ is said to be an \textit{extension} if $B = f(A)Z_B(A)$ where $Z_B(A) = \{b \in B | \forall a \in A : bf(a) = f(a)b\}$. In that case $f^{-1}(P) \subseteq \text{Spec } A$ for any $P \in \text{Spec } B$ and $\varphi : \text{Spec } B \to \text{Spec } A; P \mapsto f^{-1}(P)$ is a continuous mapping.

A monomorphic extension $f : A \hookrightarrow B$ is said to be a \textit{Zariski extension} if there exist nonempty Zariski open sets $Y(I) \subseteq \text{Spec } B$ and $X(J) \subseteq \text{Spec } A$ such that the restriction of $\varphi$ yields a homeomorphism between $Y(I)$ and $X(J)$ with their induced topologies, such that for every ideal $H \subseteq \text{rad } I$, the open set $Y(H) \subseteq \text{Spec } A$ corresponds to an open set $Y(H') \subseteq X(J)$ with $H' \subseteq \text{rad } H$. If $A \subseteq B$ is an extension, $\varphi(P) = P \cap A$. For nonempty open sets $Y(I)$ and $X(J)$, the Zariski
extension property is equivalent to \( \varphi(Y(I)) = X(J) \) and \( H = \text{rad } B(H \cap A) \)
for every radical ideal \( H \subseteq \text{rad } I \). A global Zariski extension is a Zariski
extension such that \( \varphi(I) = \text{Spec } B \).

**Proposition 2.7.** Every \( R_i \) is a global Zariski extension of its center.

**proof**

\( R_i \) is a local \( \Omega \)-ring. Hence \( \text{Spec } R_i = \{0, P'_i\} \). \( Z(R_i) \) is a discrete valuation
ring whence \( \text{Spec } Z(R_i) = \{0, p_i\} \) where \( p_i \) is the unique maximal ideal of
\( Z(R_i) \). Clearly, \( P_i' \cap Z(R_i) = p_i \) and \( P'_i \) is the unique prime ideal lying over
\( p_i \). It is trivial to see that \( \text{rad } H = \text{rad } R_i (H \cap Z(R_i)) \) for every ideal \( H \)
of \( R_i \).

**Remarks**

1) Each minimal prime ideal of \( R \) belongs to the set \( \{P_i | i \in \Lambda\} \). For suppose
\( P \) is minimal prime. Take \( 0 \neq c \in P \cap Z(R) \). We have \( R_c = \bigcap_i R_i c = \bigcap_i P_i^{n_i} \).
If \( n_1, \ldots, n_k \) are the only integers different from zero, we easily obtain
\( P_1^{n_1} \cdots P_k^{n_k} \subseteq P_1^{n_1} \cap \cdots \cap P_k^{n_k} \subseteq P \) whence \( P_i \subseteq P \) for some \( i \). Hence \( P = P_i \).

2) If each \( \sigma_i \) is a geometric kernel functor, we also have the converse result,
namely each \( P_i \) is a minimal prime ideal. It's easy to check that
\( R_i A \cap R = \{x \in R | lx \subseteq A \text{ for some } I \in \mathcal{I}^2(\sigma_i)\} \) when \( A \) is an ideal of \( R \).
Suppose \( A \) is prime and \( A \subseteq P_i \). We claim that \( R_i A \cap R = A \). If \( l x \subseteq A \) for
some \( I \in \mathcal{I}^2(\sigma_i) \), then \( l x R \subseteq A \) whence \( I \subseteq A \) or \( R \times R \subseteq A \). The inclusion
\( I \subseteq A \) leads to a contradiction because \( A \subseteq P_i \). Hence \( x \in A \).
On the other hand, \( R_i A \) is a prime ideal in \( R_i \), for suppose \( XY \subseteq R_i A \) with
\( X, Y \) ideals of \( R_i \). Then \( (X \cap R)(Y \cap R) \subseteq XY \cap R \subseteq R_i A \cap R = A \). Because
\( A \) is prime and each \( \sigma_i \) is a \( T \)-functor we obtain \( X \subseteq R_i A \) or \( Y \subseteq R_i A \).
Finally, because \( R_i A \) is prime, we have \( R_i A = P_i' \) and therefore \( A = R_i A \cap R = P_i \).
3) If the set \( \{ P_i : i \in \Lambda \} \) is equal to the set of minimal nonzero prime ideals, then a prime ideal is divisorial if and only if it is minimal prime.

For suppose \( P \) is a prime which is not minimal. There exist elements \( x_i \) such that \( x_i \in P \) but \( x_i \notin P_i \) and this is true for all \( i \). Hence

\[ R_i x_i R_i = R_i \] and certainly \( R_i P R_i = R_i \) for each \( i \). Therefore \( P \notin P_d = R \).

The other implication is trivial.

3. Arithmetical pseudo valuations on \( \Omega \)-Krull rings

Throughout this section, every \( \sigma_i \) is supposed to be geometric. In the commutative case, valuation theory is a powerful tool in studying Krull domains. The most manageable noncommutative generalization of valuation rings known to the authors is the theory of the Van Geel-primes (cfr. [13,15]). In this section we aim to relate the so called pseudo valuation functions on the set of divisor classes to primes in \( Q_{sym}(R) \). Because of its apparent importance for the definition and description of the class group, we will postpone a full account of this connexion until part II of this paper.

Let us recall some definitions. Let \( S \) be any ring. Following J. Van Geel [13,15] we will call a pair \( (P,S') \) a prime in \( S \) if and only if it satisfies the following properties:

(P1) \( : S' \) is a subring of \( S \);
(P2) \( : P \) is a prime ideal of \( S' \);
(P3) \( \forall x,y \in S : \text{if } xS'y \subset P \text{ then either } x \in P \text{ or } y \in P \).

If \( (P,S') \) is a prime in \( S \), so is \( (P,S') \) where we denote by \( S^P = \{ s \in S | sP \subset P \} \) and \( Ps \subset P \).

Primes are natural generalizations of commutative valuation rings, for, if \( S = K \) is a field, \( (P,K^P) \) is a prime in \( K \) if and only if \( K^P \) is a valuation ring in \( K \) and \( P \) is its maximal ideal.
Extending the terminology of [13] to the Ω-Krull ring case, we define:

**Definition 3.1.** An arithmetical pseudo valuation \( v \) on \( D(R) \) is a function
\[
v: D(R) \to \Gamma \cup \{\infty\}
\]
where \( \Gamma \) is a totally ordered group such that:

(V1) \( \forall I,J \in D(R): v(I \times J) = v(I) + v(J) \);

(V2) \( \forall I,J \in D(R): v(I + J) \geq \min(v(I),v(J)) \);

(V3) \( \forall I,J \in D(R): \text{if } I \subseteq J \text{ then } v(I) \geq v(J) \);

(V4) \( v(R) = 0 \) and \( v(0) = \infty \).

For any \( x \in Q = Q_{\text{sym}}(R) \) we will denote:
\[
x = \cap_{i \in \Lambda} R_i x R_i,
\]
which is clearly a divisorial \( R \)-ideal. The next theorem is a slight adaptation of a similar result for \( \Omega \)-rings (cfr [13]):

**Theorem 3.2.**

1. To any arithmetical pseudo valuation \( v \) on \( D(R) \) we can associate a prime in \( Q_{\text{sym}}(R) \).

2. To any prime \( (P,Q^P) \) in \( Q = Q_{\text{sym}}(R) \) such that \( P = \cap_{i \in \Lambda} R_i P R_i \) and \( R \subseteq Q^P \) we can associate an arithmetical pseudo valuation on \( D(R) \).

**proof**

1. Let \( v \) be an arithmetical pseudo valuation on \( D(R) \). Define
\[
P = \{x \in Q | v(C_x) > 0\}.
\]
By definition of \( v \), \( P \) is clearly a multiplicitively closed additive subgroup of \( Q_+ \) yielding that \( P \) is an ideal of \( Q^P \). If \( x, y \in Q \) such that \( x Q^P y \subseteq P \), then \( x R y \subseteq P \) because \( R \subseteq Q^P \). Therefore:
\[
0 < v(\cap_{i \in \Lambda} R_i x R_i) = v(\cap_{i \in \Lambda} R_i (\cap_{i \in \Lambda} R_i x R_i)(\cap_{i \in \Lambda} R_i y R_i)) = v(C_x x C_y) = v(C_x) + v(C_y)
\]
and thus either \( v(C_x) > 0 \) or \( v(C_y) > 0 \) yielding that \( (P,Q^P) \) is a prime in \( Q = Q_{\text{sym}}(R) \).

2. If \( (P,Q^P) \) is a prime in \( Q \) such that \( \cap_{i \in \Lambda} R_i P R_i = P \) and \( R \subseteq Q^P \), define for any divisorial \( R \)-ideal \( I \):
\[
v(I) = \{x \in Q | C_x := I \subseteq P\}.
\]
Let \( \Gamma \) be the set \( \{v(I) | I \in D(R)\} \), then \( \Gamma \) is totally
ordered by inclusion. To show this, suppose \( I, J \in D(R) \) such that both 
\( v(I) \not\subset v(J) \) and \( v(J) \not\subset v(I) \). Therefore, there exist elements \( x, y \in Q_{\text{sym}}(R) \) such that \( C_x \not\subset I \subset P \), \( C_y \not\subset J \not\subset P \), \( C_y \not\subset I \not\subset P \) and \( C_y \not\subset J \subset P \).

Because \((P, Q^P)\) is a prime, we obtain:

\[(C_x \not\subset J)Q_{\text{sym}}(R)^P(C_y \not\subset I) \not\subset P \] yielding that for some \( z \in Q^P \):

\[C_x \not\subset J \not\subset C_z \not\subset C_y \not\subset I \not\subset P \). But \( D(R) \) is an abelian group whence

\[C_x \not\subset I \not\subset C_z \not\subset C_y \not\subset J \subset P \] because for any \( z \in Q^P \) we have that \( C_z \not\subset P \subset C \) and

\[P \not\subset C_z \subset P, \] a contradiction. We claim that \( v(I) + v(J) = v(I \times J) \) is a well defined addition on \( \Gamma \) which turns \( \Gamma \) into an ordered group with unit element \( v(R) \). For if \( v(I) = v(I') \) and \( v(J) = v(J') \) for \( I, I', J \) and \( J' \in D(R) \),

then for any \( x \in v(I \times J) : C_x \not\subset I \times J \subset P \) whence \( C_x \not\subset I \subset v(J) = v(J') \),

hence \( C_x \not\subset I \times J' = C_x \not\subset J' \times I \subset P \), finally, since \( v(I) = v(I') \),

\[C_x \not\subset J' \not\subset I' \not\subset C_x \not\subset I \times J' \subset P \] follows, i.e. \( x \in v(I' \times J') \). The fact that \( v(R) \) is a unit element is obvious.

\( v(I) \leq v(J) \) yields \( v(I) + v(H) \leq v(J) + v(H) \) for any \( H \in D(R) \), for, if

\( x \in v(I \times H) \) then \( C_x \not\subset H \times I = C_x \not\subset I \times H \subset P \), i.e. \( C_x \not\subset H \subset v(I) \subset v(J) \)

whence \( C_x \not\subset J \not\subset H \subset P \). The required properties \((V1)-(V4)\) follow directly from the definition of \( v \).

4. Some examples

In this section we shall give some examples of \( \Omega \)-Krull rings.

A : A commutative Krull domain is an \( \Omega \)-Krull ring.

B : A complete matrix ring \( M_n(R) \) over an \( \Omega \)-Krull ring is itself an \( \Omega \)-Krull ring.

C : An \( \Omega \)-ring which is a global Zariski extension of its center is an \( \Omega \)-Krull ring.

D : An Azumaya algebra over a commutative Krull domain is an \( \Omega \)-Krull ring.
E : If $A$ is any simple algebra with center $K$ and $R$ is a commutative Krull domain containing $K$, then $A \otimes_K R$ is an $\Omega$-Krull ring.

proof

A and $B$ are straightforward.

C : Let $Z(R)$ be the center of $R$ and $\{P_i\}_{i \in A}$ the set of all nonzero prime ideals of $R$. Then all kernel functors $\sigma_{P_i}$ are idempotent, have property $T$ and are geometric, moreover $Q_{P_i} (R) = Z(P_i) (R)$ where $P_i = P_i \cap Z(R)$ (cfr [12,19]). Therefore it will be sufficient to prove that $R = \cap Q_{P_i} (R)$. Obviously, $R \subseteq \cap Q_{P_i} (R)$. Conversely, suppose that $c^{-1} r \in (\cap Q_{P_i} (R)) \setminus R$ where $c \in Z(R)$ and $r \in R$. Because $R$ is an $\Omega$-ring, $Rc = P_1^{k_1} \ldots P_n^{k_n}$, $RrR = P_1^{k_1} \ldots P_n^{k_n}$ where $k_i, k_i \in N$ for all $1 \leq i \leq n$ and $p_i \neq p_j$ for $i \neq j$. Therefore,

$$ Rc^{-1} rR = R^{-1} c^{-1} rR = P_1^{k_1-\ell_1} \ldots P_n^{k_n-\ell_n} \notin R. $$

Therefore, $k_i - \ell_i < 0$ for some $i$, e.g. $i=1$. Because $c^{-1} r \in Q_{P_1} (R)$ there is an ideal $I \not\subseteq P_1$ of $R$ such that $Ic^{-1} r \subseteq R$. Therefore $I = P_2^{r_2} \ldots P_s^{r_s}$ with $r_i \in N$ for all $2 \leq i \leq s$. So,

$$ Ic^{-1} rR = I.Rc^{-1} rR = P_1^{k_1-\ell_1} . P_2^{k_2-\ell_2+r_2} \ldots P_s^{k_s} \subseteq R, $$

a contradiction.

D : Let $R$ be a commutative Krull domain and $A$ an Azumaya algebra over $R$. Clearly $A \cong A^{\mathrm{opp}} \cong \mathrm{Hom}_R (R, R)$, $R$ in $R$-mod. Therefore, cfr [4], if $X^1 (R)$ is the set of prime ideals of $R$ of height one, then $A = \bigcap_{P \in X^1 (R)} Q_P (A)$. It is now easy to check that $A$ is an $\Omega$-Krull ring.

E : As in D because every ideal $I$ of $A \otimes_K R$ is of the form $A \otimes_K J$ where $J$ is an ideal of $R$.

Remark

It follows from E that any polynomial ring over a simple algebra is an $\Omega$-Krull ring giving examples of $\Omega$-Krull rings which are not Marubayashi-Krull.
5. Polynomial extensions

**Lemma 5.1.** Let \( R \) be a quasi-local \( \mathfrak{m} \)-ring with unique maximal ideal \( \mathfrak{p} \) such that \( Z(R/\mathfrak{p}) = Z(R)/\mathfrak{p} \) where \( \mathfrak{p} = \mathfrak{p} \cap Z(R) \), then \( \mathfrak{P} = Q_{p[X]}(R[X])\mathfrak{p}[X] \) is the unique maximal ideal of \( \bar{R} = Q_{p[X]}(R[X]) \).

**Proof:**

Suppose there exists an element \( x \in R[X] \setminus \mathfrak{P}[X] \) such that \( \mathfrak{P} + \bar{R} \times \bar{R} \) is a proper ideal of \( \bar{R} \). Let \( x \) be an element of minimal degree with this property, say \( x = a_nx^n + \ldots + a_0 \). First, let us assume that \( a_n \in \mathfrak{P} \). Because

\[
x' = a_{n-1}x^{n-1} + \ldots + a_0 \in R[X]\setminus \mathfrak{P}[X]
\]

with \( \deg x' < \deg x \) we can find elements \( f_i, g_i \in \bar{R} \) and \( h \in \bar{P} \) such that

\[
1 = h + \sum_i f_i x' g_i.
\]

Thus,

\[
\sum_i f_i x g_i + h = \sum_i f_i a_n x^n g_i + 1
\]

whence \( \mathfrak{P} + \bar{R} \times \bar{R} = \bar{R} \), a contradiction.

Therefore, \( a_n \in R \setminus \mathfrak{P} \). Because \( Ra_R + \mathfrak{P} = R \) we can find an element

\[
x' = x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in \mathfrak{P} + \bar{R} \times \bar{R}.
\]

We claim that \( a_i \mod \mathfrak{P} \in Z(R/\mathfrak{P}) \) for every \( 0 \leq i \leq n-1 \).

For, suppose there exists an element \( r \in R \) and an index \( i \), \( 0 \leq i \leq n-1 \) such that \( a_ir - ra_i \in R \setminus \mathfrak{P} \), then \( rx' - x'r \in R[X] \setminus \mathfrak{P}[X] \) with \( \deg (rx' - x'r) < \deg x \) and therefore \( \mathfrak{P} + \bar{R}(rx' - x'r)\bar{R} = \bar{R} \subset \mathfrak{P} + \bar{R} \times \bar{R} \), a contradiction.

Therefore we can find elements \( c_i \in C \) and \( w_i \in \mathfrak{P} \) such that \( a_i = c_i + w_i \), whence

\[
x'' = x^n + c_{n-1}x^{n-1} + \ldots + c_0 \in \mathfrak{P} + \bar{R} \times \bar{R} \text{ and } x'' \in C[X] \setminus \mathfrak{p}[X].
\]

Therefore,

\[
\mathfrak{P} + \bar{R} \times \bar{R}
\]

contains an invertible element, a contradiction. Hence \( \mathfrak{P} \) is a maximal ideal of \( \bar{R} \). Next, we have to prove that \( \mathfrak{P} \) is the unique maximal ideal.

Suppose \( Q \) were another maximal ideal of \( \bar{R} \). Let \( Q' = Q \cap R[X] \) then \( 0 \neq Q' \cap C[X] \in \text{Spec } C[X] \) and clearly \( Q' \cap C[X] \subset \mathfrak{p}[X] \). Because \( \mathfrak{p} \) is a minimal prime ideal of \( C \), so is \( \mathfrak{p}[X] \) in \( C[X] \), whence \( Q' \cap C[X] = \mathfrak{p}[X] \). This implies that \( \mathfrak{p} = Q' \cap C \).

Because \( R \) is a global Zariski extension of its center and because \( Q' \cap R \) is a nonzero prime ideal of \( R \), \( Q' \cap R = \mathfrak{p} \), yielding that \( \mathfrak{P} = \bar{R}(Q' \cap R) \subset Q \). Finally, because \( \mathfrak{P} \) is a maximal ideal of \( \bar{R} \), \( Q = \mathfrak{P} \) follows.
Using the same notation and assumptions we will prove:

**Corollary 5.2.** \( \tilde{R} \) is a local \( \Omega \)-ring.

**proof**

It will be sufficient to prove that all ideals of \( \tilde{R} \) are powers of \( \tilde{P} \). First, note that \( \tilde{P} \) is invertible in \( Q_{\text{sym}}(R[X]) \), \( (Q_{\text{sym}}(R[X])) \) exists because \( R[X] \) satisfies Formanek's condition if \( R \) does), indeed \( \tilde{P}^{-1} = \tilde{P} \cdot \tilde{P}^{-1}[X] \), where \( \tilde{P}^{-1} \) is the inverse of \( P \) in \( Q_{\text{sym}}(R) \).

Let \( I \) be a non trivial ideal of \( \tilde{R} \), then \( I \subset \tilde{P} \) and \( I\tilde{P}^{-1} \subset \tilde{R} \). By the ascending chain condition on twosided ideals of \( \tilde{R} \), either \( I = \tilde{P}^n \) for some \( n \in \mathbb{N} \), or either \( I \subset \cap \tilde{P}^n \). Because \( \cap \tilde{P}^n = 0 \) (cfr [12]) we obtain \( I = \tilde{P}^n \).

**Lemma 5.3.** Let \( S = Q_{\text{sym}}(R) \), then \( S[X] \cap \tilde{R} = R[X] \).

**proof**

Obviously, \( R[X] \subset \tilde{R} \cap S[X] \). Conversely, let \( f(X) = s_n X^n + ... + s_0 \) be any element in \( \tilde{R} \cap S[X] \), where \( s_i \in S \) for all \( 0 \leq i \leq n \). Let \( f(X) = h(X)^{-1} g(X) \) where \( g(X) \in R[X] \) and \( h(X) \in C[X] \backslash p[X] \). Because \( \overline{h(X)} = h(X) \mod P[X] \) is an element of \( Z(R[X]/P[X]) \cong Z(R/P[X]) = Z(R/P)[X] \) and \( R/P \) is a simple ring, there exists an element \( r(X) \) in \( R[X] \) such that \( \overline{h(X)}r(X) = X^m + \overline{c}_{m-1}X^{m-1} + ... + \overline{c}_0 \). Therefore, \( \overline{h(X)f(X)}r(X) = \overline{g(X)}r(X) \in (R/P)[X] \) whence \( s_n \in R/P \) and thus \( s_n \in R \). By induction, all \( s_i \in R \) whence \( f(X) \in R[X] \).

Returning to the original notation, let \( R \) be an \( \Omega \)-Krull ring with defining quasi-local \( \Omega \)-rings \( R_i \). \( P'_i \) is the maximal ideal of \( R_i \), \( C_i \) will be the center of \( R_i \), \( p'_i = P'_i \cap C_i ; P_i = P'_i \cap R, p_i = p'_i \cap Z(R) \).

As above, we will denote \( \tilde{R}_i = Q_{\text{sym}}(R[X]); \tilde{P}'_i = R_iP'_i \). Let us define a multiplicatively closed filter of ideals of \( R[X] : \mathcal{F}^2(\kappa_i) = \{I[X] \alpha | I \in \mathcal{F}^2(\sigma_i) \}; \alpha \in C_i[X] \backslash p'_i[X] \) such that \( I[X] \in R[X] \).
lemma 5.4. : $Q_{\kappa_i}^\ell (R[X]) = \bar{R}_i = Q_{\kappa_i}^r (R[X])$.

proof
Clearly, $\mathcal{L}^2(K_i)$ is a symmetric filter and $\kappa_i$ has property T. Let $g(X)^{-1}f(X) \in \bar{R}_i$ where $f(X) \in R[X]$ and $g(X) \in C_i[X] \setminus p'_i[X]$. Therefore there exists an ideal $I \in \mathcal{L}(\sigma_i)$ such that $Ig(X)$, $If(x) \in R[X]$. Thus, $Ig(X) \cdot g(X)^{-1}f(X) \subset R[X]$ yielding that $\bar{R}_i \subset Q_{\kappa_i}^\ell (R[X])$.

Conversely, suppose $g(X)^{-1}f(X) \in Q_{\kappa_i}^\ell (R[X])$ with $g(X) \in C_i[X]$ and $f(X) \in R[X]$ and $I[X] \alpha \cdot g(X)^{-1}f(X) \subset R[X]$ for some $\alpha \in C_i[X] \setminus p'_i[X]$ and $I \in \mathcal{L}^2(\sigma_i)$.

Then, $R_i[I[X] \alpha g(X)]^{-1}f(X) = R_i[X]g(X)^{-1}f(X) \subset R_i[X]$ whence $g(X)^{-1}f(X) \in R_i[X] \alpha^{-1}$.

Thus, $Q_{\kappa_i}^\ell (R[X]) = \bar{R}_i$.

Analogously one proves that $Q_{\kappa_i}^r (R[X]) = \bar{R}_i$.

If $R$ is an $\Omega$-Krull ring and if $S = Q_{\text{sym}}(R)$, then it is easy to check that $S[X]$ is an $\Omega$-ring (cfr [12,1]). Let $(M_j)_j \in J$ be the set of all its nonzero prime ideals, let $m_j = Z(S)[X] \cap M_j$ and define a symmetric idempotent filter as follows :

$\mathcal{L}^2(\omega_j) = \{I[X] \alpha | \alpha \in \mathcal{L}^2(\omega_j), \alpha \in Z(S)[X] \setminus m_j \text{ such that } I[X] \alpha \subset R[X]\}$.

lemma 5.5. : $Q_{\omega_j}^\ell (R[X]) = Q_{m_j}^\omega (S[X]) = Q_{\omega_j}^r (R[X])$.

proof
Along the lines of lemma 5.4.

Theorem 5.6. : Let $R$ be an $\Omega$-Krull ring such that $Z(R_i/p'_i) = C_i/p'_i$ for all $i \in \Lambda$, then $R[X]$ is an $\Omega$-Krull ring.

proof
Let $R = \bigcap_{i \in \Lambda} R_i$. By Corollary 5.2., $\bar{R}_i$ is a quasi-local $\bar{\Omega}$-ring and $\bar{R}_i \cap S[X] = R_i[X]$ by lemma 5.3. Because $S[X]$ is an $\Omega$-ring which is a global Zariski extension of its center, we obtain from the proof of section 4.c. that
$S[X] = \bigcap_{j \in J} Q_{m_j}(S[X])$. Moreover,

$$R[X] = \bigcap_{i \in A} R_i[X] = \bigcap_{i \in A} (\bar{R}_i \cap S[X]) = (\bigcap_{i \in A} \bar{R}_i) \cap \bigcap_{j \in J} Q_{m_j}(S[X]).$$

From lemmas 5.4. and 5.5. we know that the rings $\bar{R}_i$ and $Q_{m_j}(S[X])$ are over-rings of $R[X]$ satisfying the requirements of the definition of $\Omega$-Krull ring.

Finally, let us verify condition 4 of definition 1.2. Let $f(X) \in R[X]$.

The ideal $R[X]f(X)R[X]$ contains a central element, say $g(X) = a_nX^n + \ldots + a_0$.

Then $Ra_kR \in L^2(\sigma_i)$ for almost all $i \in A$, yielding that $\bar{R}_i g(X) = \bar{R}_i$ for all but finitely many $i \in A$.

Also, $S[X]g(X) \in L^2(\mathbb{Z}(S[X]) - m_j)$ for almost all $j \in J$ (because $S[X]$ is an $\Omega$-ring) implying that $Q_{m_j}(S[X])g(X) = Q_{m_j}(S[X])$ for all but finitely many $j$.

This completes the proof.

Remark

It is not known to the authors whether the condition $Z(R_i/P_i^j) = C_i/p_i^j$ can be dropped in general. We conjecture that this is not the case. An other intriguing question is whether $R[T_1, \ldots, T_n]$ remains $\Omega$-Krull if $R[T]$ is $\Omega$-Krull.

Acknowledgement

The authors wish to thank Professor F. Van Oystaeyen for making several helpful suggestions.
References


