TOWARDS A NONCOMMUTATIVE VERSION OF THE
BASS - TATE SEQUENCE

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0. INTRODUCTION

In [9], S-Krull domains were introduced as intersections of (strongly) serial subrings of a skewfield satisfying a finite character type property. These rings are very special examples of Marubayashi- and Chamarie Krull rings. If they satisfy a polynomial identity, their arithmetic is closely related to that of its center (in fact, they are Zariski central rings, cfr. [9]). In the general case however, there remain some intriguing problems, e.g. if every ideal intersects the center nontrivially, do minimal prime ideals satisfy the unique lying over property with respect to minimal primes in the center? This paper is a first attempt to study the relation between the arithmetical and $K$-theoretical invariants of these rings.

In the first section we provide an intrinsic characterization of S-Krull domains, stating that they are exactly those Marubayashi-Krull (*) : The author is supported by an NFWO/FNRS - grant (Belgium)
rings for which every element is normalizing. Further on, we restrict attention to the one dimensional case, i.e. to so called S-Dedekind domains. In section two we show that the Grothendieck group of an S-Dedekind domain \( R \) is isomorphic to \( \mathbb{Z} \oplus \text{Cl}(R) \), where \( \text{Cl}(R) \) denotes the (normalizing) class group of \( R \). The proof touches upon Dieudonné's theory of noncommutative determinants. In the third section we establish an isomorphism between the class and Picard group and a cohomological interpretation of this Picard group: \( \text{Pic}(R) \cong H^1(X, \mathcal{U}(R))/\{\mathbb{Z}^\times, \mathbb{Z}^\times\} \).

The main theorem of this paper is contained in the final section. We present a noncommutative version of the famous Bass-Tate sequence for Dedekind domains which states, briefly, that the K-groups of a Dedekind domain are determined by the K-groups of its residue fields and of its field of fractions. The proof of this theorem is very similar to the commutative one and it depends on a noncommutative version of Matsumoto's theorem, due to S. Green (5) and on the existence of Mennicke symbols for S-Dedekind domains. Exactness of the sequence \( \oplus K_1(R/P) \) is not included because it depends on the validity of a noncommutative analogue of the Kubota-Bass theorem, a problem which we hope to attack in a subsequent paper.
1. A CHARACTERIZATION OF S-KRULL DOMAINS

Throughout this note, \( R \) will always be a prime Goldie ring with simple Artinian (left and right) ring of quotients \( Q \).
Recall that a ring \( S \) is said to be left serial (resp. left strongly serial) if the set of its left ideals is totally ordered (resp. well ordered). It was proved in \((\mathcal{G})\) that whenever \( R \) has a left serial overring \( S \) in \( Q \), \( R \) is a domain and \( S \) is a total subring of the skewfield \( Q \), i.e. for every \( x \in Q^* \), either \( x \in S \) or \( x^{-1} \in S \). Recall that a total subring of a skewfield \( Q \) is a valuation ring of \( Q \) if and only if it is invariant under inner automorphisms of \( Q \) and moreover valuation rings of \( Q \) correspond bijectively to valuation functions \( v : Q^* \rightarrow \Gamma \) for some ordered group \( \Gamma \). If \( \Gamma = \mathbb{Z} \), then \( v \) is called a principal valuation and the corresponding valuation ring \( O_v \) of \( Q \) is said to be a principal valuation ring. Recall from \((\mathcal{G})\) that an overring \( S \) of \( R \) in \( Q \) is a principal valuation ring of \( Q \) if and only if it is left strongly serial.

**Definition 1.1:** \( R \) is a left (resp. right) S-Krull domain if there is a family of overrings of \( R \) in \( Q \), \( \{ O_v : v \in V \} \) say, such that the following conditions are met:

- **K 1:** for every \( v \), \( O_v \) is left (resp. right) strongly serial
- **K 2:** \( R = \cap O_v \) (we may suppose that \( O_v \neq Q \) for all \( v \))
- **K 3:** for every element \( r \) of \( R \), the set \( E(r) = \{ v \in V : r \text{ is not a unit in } O_v \} \) is finite.

From \((\mathcal{G})\) we retain that this definition is left-right
invariant.

If \( A, B \subseteq \mathbb{Q} \), write \( (A; R) = \{ x \in \mathbb{Q} : xA \subseteq B \} \), \( (A; R) = \{ x \in \mathbb{Q} : Ax \subseteq B \} \). A (nonzero) left (resp. right) fractional \( R \)-ideal is a nonzero left (resp. right) \( R \)-submodule \( A \) of \( \mathbb{Q} \) such that there exists an element \( q \) in \( \mathbb{Q} \) such that \( Aq \subseteq R \) (resp. \( qA \subseteq R \)). \( A \) is said to be a left (resp. right) divisorial \( R \)-ideal if it is left (resp. right) fractional and if \( (A; R):R = A \) (resp. \( (A; R):R = A \)).

If \( R \) is an \( S \)-Krull domain, then every left divisorial \( R \)-ideal is right divisorial, in particular, every element \( q \) of \( \mathbb{Q} \) is \( R \)-normalizing, i.e. \( qR = Rq \).

Furthermore, the divisorial ideals form a group under the multiplication \( A \star B = (AB; R):R \) and the set of divisorial \( R \)-ideals contained in \( R \) satisfies the ascending chain condition.

An \( S \)-Krull domain \( R \) is a maximal order in the sense of (10) i.e. there exists no proper overring \( S \) of \( R \) in \( \mathbb{Q} \) such that \( qSq'C \subseteq R \) for some elements \( q, q' \) in \( \mathbb{Q} \).

We are now ready to prove the following characterization theorem for \( S \)-Krull domains.

**Theorem 1.2:** A prime Goldie ring \( R \) with classical ring of quotients \( Q \) is an \( S \)-Krull domain if and only if:

1. every element \( q \) of \( Q \) is \( R \)-normalizing
2. the divisorial ideals contained in \( R \) satisfy the acc.
3. \( R \) is a maximal order.

**Proof**

In view of the cited results from (9), one implication is
obvious. Conversely, let us assume that $R$ satisfies the properties $K_1$ to $K_3$. Because every element is normalizing, $R$ is clearly a domain. Further, one can prove as in (7) that the set of divisorial $R$-ideals, $\mathcal{D}(R)$, equipped with the multiplication $A \star B = (AB:R):R$, is an abelian group. Now, $\mathcal{D}(R)$ is an ordered group for the ordering: $A \preceq B$ iff $A \supset B$. It is readily verified that each finite nonempty subset of $\mathcal{D}(R)$ has a supremum, $A_1 \cap A_2 \cap \ldots \cap A_n$, and an infimum, $(A_1^+ \ldots + A_n:R):R$. Moreover, condition $K_2$ states that any nonempty subset of positive elements of $\mathcal{D}(R)$ (i.e. divisorial $R$-ideals contained in $R$) has a minimal element. A well-known theorem on commutative ordered groups satisfying these properties (cfr. (??)) yields that:

$\mathcal{D}(R) \cong \mathcal{Z}(A)$ for some index set $A$ and this isomorphism is order preserving. Of course, the order relation on $\mathcal{Z}(A)$ is defined by: $(a_i)_{i \in A} \preceq (b_i)_{i \in A}$ iff $a_i \leq b_i$ for all $i \in A$. Let $f: \mathcal{D}(R) \to \mathcal{Z}(A)$ be such an order preserving isomorphism. Put $e_i = (\delta_{ij})_{j \in A}$ and let $P_i = f^{-1}(e_i)$.

Thus, any element $A$ of $\mathcal{D}(R)$ can be written uniquely as $A = P_1^{n_1} \star \ldots \star P_k^{n_k}$ ($n_i \in \mathbb{Z}$). We claim that every $P_i$ is a complete prime ideal of $R$ (i.e. $xy \in P_i$ iff $x$ or $y$ in $P_i$).

For, if $xy \in P_i$, then $Rx \star Ry = Rxy \subset P_i$. Further, $f(Rx) = \Sigma n_j e_j$ and $f(Ry) = \Sigma m_j e_j$ where $n_j, m_j \geq 0$. In particular, $f(Rx) + f(Ry) = \Sigma (n_j + m_j)e_j \geq f(P_i) = e_i$. Therefore, $n_i \geq 1$ or $m_i \geq 1$ yielding that either $x \in P_i$ or $y \in P_i$.

Now, let $P$ be a height one prime ideal of $R$ and $0 \neq x \in P$ then $P_1^{n_1} \ldots P_k^{n_k} \subset (P_1^{n_1} \ldots P_k^{n_k}:R):R = Rx \subset P$ whence $P = P_i$ for some $i$. Conversely, if $P$ is a prime generator of $\mathcal{D}(R)$ and if there exists an ideal $Q$ of $R$ such that $Q$ is
prime and $0 \neq Q \subset P$. Then, $P_i \subset Q \subset P$ whence $f(P) < f(P_i) = e_i$, a contradiction. Thus, $D(R)$ is generated by the height one prime ideals of $R$. Define for every $i \in I$ a mapping $w_i : D(R) \to \mathbb{Z}$; $A = P_1^{n_1} \cdots P_k^{n_k} \to n_i$. An easy (but not entirely trivial) computation shows that these $w_i$ induce principal valuations $v_i$ on $Q$ by: $v_i(q) = w_i(Rq)$. For every $i$, $\mathcal{O}_{v_i} = \{q \in Q : v_i(q) > 0\}$ is a principal valuation ring and $R = \cap v_i$, for, if $q \in \cap v_i$, then $Rq = P_1^{n_1} \cdots P_k^{n_k} \subset R$ because all $n_i > 0$. Finally, $E(q)$ is finite for every $q$ because there are only a finite number of nontrivial terms in the decomposition of $Rq$, finishing the proof.

2. $K_0$ AND CLASS GROUP OF S-DEDEKIND DOMAINS

From now on, we will mainly be interested in the one-dimensional case, i.e.

**Definition 2.1.** An $S$-Krull domain $R$ is said to be $S$-Dedekind if every (nonzero) prime ideal of $R$ is maximal.

In view of Th. 1.2., the following characterization is easy to prove:

**Proposition 2.2.** A prime Goldie ring $R$ with classical ring of quotients $Q$ is an $S$-Dedekind domain iff:
1. every element $q$ of $Q$ is $R$-normalizing
2. $\mathbb{F}(R)$, the set of all fractional $R$-ideals, is a group under multiplication.
It follows from the proof of Th 1.2. that \( F(R) \) is the free abelian (!) group generated by the nonzero prime ideals of \( R \). Thus, every nonzero ideal of \( R \) can be expressed uniquely as a (finite) product of maximal ideals. This yields:

**Lemma 2.3.** : For any nonzero ideal \( I \) of an S-Dedekind ring \( R \), the quotient \( R/I \) is a principal ideal ring.

**proof**

There are only a finite number of distinct maximal ideals in \( R/I \) containing \( I \), say \( M_1, \ldots, M_k \). We will first show that each \( M_i \) is a principal ideal modulo \( I \). Let \( x \) be an element of \( M_1 \setminus M_1^2 \). Since the ideals \( M_1^2, M_2, \ldots, M_k \) are pairwise relative prime, it follows that there is an element \( y_1 \) of \( R \) such that: \( y_1 \equiv x \mod M_1^2 \) and \( y_1 \equiv 1 \mod M_j \) for \( j > 1 \), using a noncommutative version of the Chinese remainder theorem (cfr. e.g. ([M3])). The ideal generated by \( I \) and \( y_1 \) is contained in \( M_1 \), but is not contained in \( M_1^2 \) or in any other maximal ideal, thus it can only be \( M_1 \) itself. This proves that \( M_1 \) is a principal ideal modulo \( I \). But every ideal of \( R/I \) is a product of maximal ideals, so this completes the proof.

**Definition 2.4.** : Two nonzero ideals \( A \) and \( B \) in an S-Dedekind domain \( R \) belong to the same ideal class if there exist nonzero elements \( x \) and \( y \) in \( R \) such that \( Ax = By \).

Clearly, the ideal classes of \( R \) form an abelian group under multiplication, with the class of principal ideals as identity element. We will use the notation \( Cl(R) \) for the ideal
class group of $R$, and the notation $(A) \in Cl(R)$ for the ideal class of $A$. Note that $(A) = (B)$ if and only if $A$ is isomorphic to $B$ as left $R$-modules.

**Lemma 2.5.** Given nonzero ideals $A$ and $B$ in an $S$-Dedekind domain $R$, there exists an ideal $A'$ in the ideal class of $A$ which is prime to $B$.

**Proof**
Choose a nonzero element $a$ of $A$ and define the ideal $C$ by the equation: $CA = Ra$. Applying lemma 2.3. to the ideal $C$ modulo $BC$ we see that $C = BC + Rx$ for some element $x$ of $R$. Multiplying this equation with $A$ and then dividing by $a$ we obtain: $R = B + Ax^{-1}$, finishing the proof.

**Lemma 2.6.** If $A$ and $B$ are nonzero ideals in an $S$-Dedekind domain $R$, then the left $R$-module $A \otimes B$ is isomorphic to $R \otimes AB$.

**Proof**
In view of the foregoing lemma we may assume that $A$ and $B$ are relatively prime, i.e. $A + B = R$. Map $A \otimes B$ onto $R$ by the correspondence $a \otimes b \mapsto a + b$. The kernel is isomorphic to $A \cap B = AB$ and because $R$ is projective, $A \otimes B \cong R \otimes AB$.

**Proposition 2.7.** Every ideal in an $S$-Dedekind domain $R$ is a finitely generated left $R$-module. Conversely, every finitely generated projective left $R$-module is isomorphic to a direct sum of ideals.
proof

If $A$ is a nonzero ideal, $A^{-1} \cdot A = R$ whence there are elements $a_1$ in $A$, $f_1$ in $A^{-1} = \text{Hom}_R(A, R)$ such that $\Sigma f_1 a_1 = 1$, whence for every $a \in A$, $a = \Sigma (af_1)a_1 \subset Ra_1 + \ldots + Ra_k$ and $A$ is thus f.g. projective by the dual basis theorem. Conversely, any finitely generated projective left $R$-module $P$ can be embedded in a free left $R$-module $R^k$ for some $k$. Projecting to the $k$-th factor we obtain a homomorphism $\gamma : P \to R$ with $\text{Ker}(\gamma) \subset R^{k-1}$. Since the image of $P$ in $R$ is a left ideal whence an ideal and thus projective, we have $P \cong \text{Ker}(\gamma) \oplus \text{Im}(\gamma)$ and an induction argument finishes the proof.

Recall that the Grothendieck group $K_0(R)$ is an additive group defined by generators and relations as follows. There is to be one generator $[P]$ for each isomorphism class of finitely generated projective modules $P$ over $R$ and one relation: $[P] + [Q] = [P \oplus Q]$ for each pair of f.g. projective left $R$-modules. As in the commutative case we aim to prove:

Theorem 2.8. If $R$ is an S-Dedekind domain, then:

$K_0(R) = \mathbb{Z} \oplus \text{Cl}(R)$.

proof

In view of lemma 2.6. and proposition 2.7. it will be sufficient to check that the map $A_1 \oplus \ldots \oplus A_k \to (r, (A_1 \ldots A_k))$ sends isomorphic left $R$-modules to the same element in $\mathbb{Z} \oplus \text{Cl}(R)$, i.e. we have to prove a noncommutative version of Steinitz's theorem.
Theorem 2.9. (Steinitz, noncommutative) Two direct sums \( A_1 \oplus \cdots \oplus A_r \) and \( B_1 \oplus \cdots \oplus B_s \) of nonzero ideals of \( R \) are isomorphic as left \( R \)-modules if and only if \( r = s \) and \( \langle A_1, \ldots, A_r \rangle = \langle B_1, \ldots, B_r \rangle \).

proof

First, it is trivial to check that any left \( R \)-module morphism \( \phi : A \to B \) between two nonzero ideals is completely determined by an element \( q \in Q \) such that \( \phi(a) = aq \) for every \( a \in A \). Similarly, for any left \( R \)-module morphism \( \phi : A_1 \oplus \cdots \oplus A_r \to B_1 \oplus \cdots \oplus B_s \) there exists a unique \( r \times s \) matrix \( M = (m_{ij}) \) with entries in \( Q \) such that \( \phi(a_1, \ldots, a_r) = (b_1, \ldots, b_s) \) with \( b_i = \sum a_j m_{ji} \) for every \( (a_1, \ldots, a_r) \in A_1 \oplus \cdots \oplus A_r \). If \( \phi \) is an isomorphism, then \( M \) has an inverse \( M^{-1} \) whence \( r = s \).

In order to prove \( \langle A_1, \ldots, A_r \rangle = \langle B_1, \ldots, B_r \rangle \), we note first that \([Q^*, Q^*] \subseteq U(R) \). This follows immediately from the fact that \( F(R) \) is an abelian group. Using this fact it is now easy to verify that the Dieudonné-determinant (cfr. e.g. (3)) of an invertible matrix \( N \) in \( M_k(Q) \), such that the \( i \)-th column of \( N \) consists of elements in \( C_i \) for some \( R \)-ideal \( C_i \), belongs to \( \pi((C_1, \ldots, C_k)^*) \) where \( \pi : Q^* \to (Q^*)^{ab} \).

Applying this to the determinant of the product matrix:

\[
\begin{bmatrix}
a_1 & 0 & \cdots & 0 \\
0 & a_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_r
\end{bmatrix}
\]

where \( (a_1, \ldots, a_r) \in A_1 \oplus \cdots \oplus A_r \), we get: \( \pi(a_1, \ldots, a_r) \cdot \det(M) \in \pi(B_1, \ldots, B_r) \), whence there exists an element \( q \in \pi^{-1}(\det(M)) \) such that \( A_1 \cdots A_r \cdot q \subseteq B_1 \cdots B_r \) (because \([Q^*, Q^*] \subseteq U(R) \)). Similarly, there exists an element \( q' \in \pi^{-1}(\det(M^{-1})) \) such that \( B_1 \cdots B_r \cdot q' \subseteq A_1 \cdots A_r \).

Finally, because \( qq' \in U(R) \) we obtain that \( A_1 \cdots A_r = A_1 \cdots A_r \cdot qq' \subseteq B_1 \cdots B_r \cdot q' \subseteq A_1 \cdots A_r \), whence \( B_1 \cdots B_r \cdot q' = A_1 \cdots A_r \).
Thus, $\langle A_1 \ldots A_r \rangle = \langle B_1 \ldots B_s \rangle$. The converse implication follows from Lemma 2.6.

3. A COHOMOLOGICAL INTERPRETATION OF $\text{Cl}(R)$

For a commutative Dedekind domain $D$ it is well known that there is a natural isomorphism between $\text{Cl}(R)$ and $\text{Pic}(D)$, the Picard group of $D$. This fact may for example be proved by establishing an isomorphism between the Weil divisors and the Cartier divisors on the affine scheme $\mathcal{X} = \text{Spec}(D)$. Furthermore, $\text{Pic}(R) \cong H^1(X, U(\mathcal{R}))$ (cf. e.g. [12]). In this section we aim to give an analogous cohomological interpretation of $\text{Cl}(R)$ for an $S$-Dedekind domain $R$.

Let $X = \text{Spec}(R)$ be the prime spectrum of $R$ equipped with the usual Zariski topology (cf. e.g. [12]). Because every element of $R$ is normalizing, the sets $X(r) = \{ P \in \text{Spec}(R) : r \notin P \}$ form a basis for this topology.

We will define two sheaves of rings on $X$. The first, $\mathcal{K}$, is the constant sheaf corresponding to $\mathbb{Q}$. The second, $\mathcal{R}$, is the usual noncommutative affine scheme introduced by F. Van Oystaeyen in [12] using symmetric localization. In this special case, $\Gamma(\mathcal{R}, X(r)) = \{ x r^{-n} : x \in R, n \in \mathbb{N} \}$ whereas the stalks are $R_p$, the classical localization at the left and right Ore-set $R-P$. It is clear that $\mathcal{R}$ is a sheaf of $S$-Dedekind domains. We will mainly be interested in the following two sheaves of not necessarily abelian groups: $U(\mathcal{R})$ is defined by $\Gamma(U(\mathcal{R}), X(r)) = U(\Gamma(\mathcal{R}, X(r)))$ where $\Gamma(\cdot)$ denotes 'taking units' and restriction morphisms are inclusions; $\mathcal{N}(\mathcal{R})$, the sheaf of $\mathcal{R}$-normalizing elements, is just the constant sheaf $\mathbb{Q}^\ast$. Finally, remark that $U(\mathcal{R})$ is a normal subsheaf of $\mathcal{N}(\mathcal{R})$. 
Definition 3.1.: A Cartier divisor on $X$ is a global section of the sheaf $N(R)/U(R)$. Thinking of the properties of quotient sheaves, we see that a Cartier divisor on $X$ can be described by giving an open cover \{V_i\} of $X$, and for each $i$ an element $s_i \in \Gamma(N(R), V_i)$ such that for all $i$ and $j$, $s_i s_j^{-1} \in \Gamma(U(R), V_i \cap V_j)$.

A Cartier divisor is said to be principal if it is an image of the natural map $\Gamma(N(R), X) \to \Gamma(N(R)/U(R), X)$. Two Cartier divisors are linearly equivalent if their quotient is principal. The group of Cartier divisor classes will be denoted by $\text{Pic}(R)$.

The following proposition is a special case of Prop. 4.4. of (7) using the fact that $X$ is locally factorial (i.e. $\text{Cl}(R_p) = 1$ for every $p \in X$):

Proposition 3.2.: If $R$ is an S-Dedekind domain, then $\text{Cl}(R) \approx \text{Pic}(R)$.

We have the following exact sequence of non-abelian sheaves:

$1 \to U(R) \to N(R) \to N(R)/U(R) \to 1$ and $\text{Pic}(R) = \text{coker}(\Gamma(N(R), X) \to \Gamma(N(R)/U(R), X))$

whence $\text{Pic}(R) = \ker(\Gamma^1(X, U(R)) \to \Gamma^1(X, N(R)))$ where the $\Gamma^1(.)$ are the non-abelian cohomology pointed sets of (14). Luckily, one can also give an interpretation of $\text{Pic}(R)$ in terms of usual (i.e. abelian) Čech-cohomology:

Proposition 3.3.: If $R$ is an S-Dedekind domain, $\text{Pic}(R) \approx \Gamma^1(X, U(R)[(\mathbb{Q}_*^\times, \mathbb{Q}_*^\times)]$)

proof

Because fractional ideals commute in an S-Dedekind domain, $N(R)/U(R)$ is a sheaf of abelian groups. Therefore we have the exact diagram:

\[
\begin{array}{ccccccccc}
\text{1} & \to & U(R) & \to & N(R) & \to & N(R)/U(R) & \to & \text{1} \\
& & \downarrow\phi & & & & \downarrow & & \\
\text{1} & \to & \text{Ker}\phi & \to & N(R)^{ab} & \to & N(R)/U(R) & \to & \text{1}
\end{array}
\]
where $N(R)^{ab}$ denotes the abelianized sheaf of $N(R)$, i.e. the constant (1) sheaf with sections $(Q^*)^{ab}$. By local consideration, one verifies easily that $\text{Ker}(\phi) = U(R)/[\mathbb{Q}^*, \mathbb{Q}^*]$, thus we get a long exact sequence using Čech - cohomology:

$$1 \to \Gamma(U(R)/[\mathbb{Q}^*, \mathbb{Q}^*], X) \to \Gamma(N(R)^{ab}, X) \xrightarrow{\alpha} \Gamma(N(R)/U(R), X) \xrightarrow{\beta} \check{H}^1(X, U(R)/[\mathbb{Q}^*, \mathbb{Q}^*]) \to \check{H}^1(X, N(R)^{ab}) \to$$

Now, clearly $\text{Pic}(R) = \text{Ker}(\beta) = \text{Coker}(\alpha)$ and finally $\check{H}^1(X, N(R)^{ab}) = 1$ for $N(R)^{ab}$ is a constant sheaf, finishing the proof.

4. TOWARDS A BASS - TATE SEQUENCE FOR S-DEDEKIND DOMAINS

In this section we aim to prove the following:

**Theorem 4.1.** If $R$ is an S-Dedekind domain, then the sequence:

$$K_2(Q) \to \oplus K_1(R/P) \to K_1(R) \to K_1(Q) \to \oplus K_0(R/P) \to K_0(R) \to K_0(Q)$$

where the direct sums are taken over all prime ideals of $R$, is a complex of abelian groups which is exact except perhaps in $\oplus K_1(R/P)$.

**Remark:** Exactness in $\oplus K_1(R/P)$ depends on the validity of a noncommutative version of the Kubota-Bass theorem. This problem will be treated in ($\mathcal{E}$).

Let us briefly recall the definition of the functors $K_1(.)$ and $K_2(.)$:

For any ring $R$ let $\text{GL}_n(R)$ denote the general linear group consisting of all $n \times n$ invertible matrices over $R$, and let $\text{GL}(R)$ denote the direct limit of the sequence: $\text{GL}_1(R) \subset \text{GL}_2(R) \subset \text{GL}_3(R) ...$ where each $\text{GL}_n(R)$ is injected into $\text{GL}_{n+1}(R)$ by adjunction of 1 on the diagonal. A matrix in $\text{GL}(R)$ is called elementary if it coincides with the identity matrix except for a single off-diagonal entry. By Whitehead's lemma, $E(R)$, the subgroup of $\text{GL}(R)$ generated by all elementary matrices is precisely the
commutator subgroup of \( GL(R) \). The abelian quotient group \( GL(R)/E(R) \) is called the Whitehead group \( K_1(R) \) of \( R \).

If \( Q \) is a skewfield, it follows from the construction of the Dieudonné determinant and from the definition of \( K_1(.) \) that \( K_1(Q) \cong Q^{\times \cdot ab} \), the (multiplicative) group of nonzero elements of \( Q \). Abelianised

If \( R \) is an S-Dedekind domain, it is possible to define \( SL_n(R) \) to be the (normal) subgroup of \( GL_n(R) \) consisting of matrices with Dieudonné determinant \( \pi(1) \), \( SL(R) \) is the direct limit of the \( SL_n(R) \). With \( E^{r}_{ij} \), we will denote the elementary matrix with \((i,j)\) entry equal to \( r \).

**Proposition 4.2.** (Bass, noncommutative) If \( R \) is an S-Dedekind domain, every matrix in \( GL(R) \) (resp. \( SL(R) \)) can be reduced by elementary row and column operations to a matrix in the subgroup \( GL_2(R) \) (resp. \( SL_2(R) \)).

**proof**

Let \((a_1, \ldots, a_n)\) be the last row of an arbitrary matrix \( A \) in \( GL_n(R) \), with \( n \geq 3 \). There exist elements \( b_1 \) in \( R \) such that \( a_1 b_1 + \ldots + a_n b_n = 1 \).

If \( a_2 = 0 \), \( A' = A.E^{b_1}_{1,2}, E^{b_2}_{2,2}, \ldots E^{b_n}_{n,2} \) then \( A'(n,2) = 1 \).

If \( a_2 \neq 0 \), there are only a finite number of maximal ideals containing \( a_2, a_3, \ldots, a_n \); say \( M_1, \ldots, M_s \). Suppose that the first \( r \) of these ideals contain \( a_1 \) and the remaining do not. By the Chinese remainder theorem one can find an element \( e \) of \( R \) such that:

\[
e \equiv 1 \mod M_1, M_2, \ldots, M_r \quad \text{and} \quad e \equiv 0 \mod M_{r+1}, \ldots, M_s
\]

\[
A' = A.E^{e}_{n,1}
\]

has last row \((a_1 + a_2 e, \ldots, a_n)\).

Therefore, we may suppose that \( A \) has last row \((a_1, \ldots, a_n)\) such that

\[
Ra_1 + \ldots + Ra_{n-1} = R. \quad \text{Now, if } a_1 c_1 + \ldots + a_{n-1} c_{n-1} = 1 - a_n, \quad \text{then}
\]

\[
A' = A.E^{c_1}_{1,n} \ldots E^{c_{n-1}}_{n-1,n}
\]

is such that \( A'(n,n) = 1 \).

Then \( B = A'.E^{a_1}_{n,1} \ldots E^{a_{n-1}}_{n,n-1} \) has last row equal to \((0,0, \ldots, 0, 1)\) and last column, say \((d_1, \ldots, d_{n-1}, 1)\). Finally, \( C = E^{d_1}_{1,n} \ldots E^{d_{n-1}}_{n-1,n} B \) lies
in the subgroup $GL_{n-1}(R)$. Continuing in this manner, $A$ will be congruent to a matrix in $GL_2(R)$. A similar argument may be applied to $SL(R)$.

**Remark:** If $A \in K_1(R)$ is determined by the class of the matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then $\det(A) \in \pi(U(R))$ because $\det(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = ad - ac^{-1}b$ if $a \neq 0$ and $= -bc$ if $a = 0$.

For $n \geq 3$, the Steinberg group $St_n(R)$ is the group defined by generators $X_{ij}^r$ (where $i$ and $j$ range over all pairs of distinct integers between 1 and $n$ and $r$ ranges over $R$) subject to the relations:

$$X_{ij}^r X_{ij}^{-1} = X_{ij}^{r+s} ; [X_{ij}^r, X_{jk}^s] = X_{kl}^{rs} \text{ if } i \neq 1 \text{ and } [X_{ij}^r, X_{kj}^s] = 1 \text{ if } j \neq k \text{ and } i \neq 1.$$ 

There exists a canonical homomorphism $\phi : St_n(R) \to GL_n(R)$ assigning $E_{ij}^r$ to $X_{ij}^r$. The Milnor group $K_2(R)$ is defined to be $\text{Ker}(\phi)$.

To begin the proof of Th.4.1, let $K_0(R/P) \to K_0(R)$ be the homomorphism which sends the standard generator $[R/P] (K_0(R/P) \cong \mathbb{Z}$ because $P$ is a complete prime ideal of $R$) whence $R/P$ is a skewfield) to the difference $[R] - [P]$ in $K_0(R)$. Let $Q^* \to K_0(R/P)$ be the homomorphism which sends each nonzero field element to the sum $\sum v_p(x) [R/P]$, where $v_p$ is the discrete valuation on $Q$ corresponding to $P$. Clearly, this morphism factors through $K_1(Q) \cong \mathbb{Q}^* ab$. Exactness of the resulting sequence $K_1(R) \to K_1(Q) \to K_0(R/P) \to K_0(R) \to K_0(Q) \to 1$ is left as an easy exercise to the reader (the morphism $K_1(R) \to K_1(Q)$ is, of course, taking the Dieudonné-determinant).

For commutative Dedekind domains one uses Mannicke symbols in order to define the homomorphism $K_1(R/P) \to K_1(R)$. Our first task will be to extend this construction to $S$-Dedekind domains.

Let $a$ and $b$ be relative prime elements of $R$, i.e. $aR + Rb = R$, then there exist elements $c$ and $d$ such that $ad - cb = 1$. Now, consider the matrix:
\[ A = \begin{bmatrix} a & b \\ a^{-1}c & d \end{bmatrix} \] We claim that \( A \in \text{SL}_2(R) \), for, if \( c \neq 0 \), then:
\[ A^{-1} = \begin{bmatrix} a^{-1}c^{-1} & -a^{-1}b \\ -c & a \end{bmatrix} \] which belongs to \( M_2(R) \) because \([Q^*,Q^*] \in U(R) \) and if \( c = 0 \), then:
\[ A^{-1} = \begin{bmatrix} d & -a^{-1}b \\ 0 & a \end{bmatrix} \in M_2(R) \]. Finally, calculating the Dieudonné-determinant yields that \( A \in \text{SL}_2(R) \).

If we work modulo the normal subgroup \( E(R) \), we claim that this matrix depends only on \( a \) and \( b \). For, if \( c' \) and \( d' \) are other elements of \( R \) such that \( ad' - c'b = 1 \), then:
\[
\begin{bmatrix} a & b \\ a^{-1}c' & d' \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ \ast & 1 \end{bmatrix} \in E(R)
\]

**Definition 4.3.** The Mennicke symbol \( \frac{b}{a} \) is defined to be the element of the subgroup \( SK_1(R) = \text{SL}(R)/E(R) \subset K_1(R) \) which is represented by the unimodular matrix \( \begin{bmatrix} a & b \\ a^{-1}c & d \end{bmatrix} \) whenever \( a \) and \( b \) are relative prime.

**Lemma 4.4.** The Mennicke symbol \( \frac{b}{a} \in K_1(R) \) is symmetric, bimultiplicative, and is not altered if we add a multiple of \( a \) to \( b \) or a multiple of \( b \) to \( a \).

**Proof**
The properties \( \frac{b}{a} = \frac{b + ra}{a} \) and \( \frac{b}{a} = \frac{b}{a + sb} \) follow from the equations:
\[
\begin{bmatrix} a & ra+b \\ \ast & \ast \end{bmatrix} = \begin{bmatrix} a & b \\ a^{-1}c & d \end{bmatrix} \begin{bmatrix} 1 & a^{-1}ra \\ 0 & 1 \end{bmatrix}
\]
\[
\begin{bmatrix} a+sb & b \\ \ast & \ast \end{bmatrix} = \begin{bmatrix} a & b \\ a^{-1}c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b^{-1}sb & 1 \end{bmatrix}
\]

If \( u \) is a unit of \( R \), then:
\[
\frac{u}{a} = \frac{u}{1} = \frac{0}{1} = 1 \in K_1(R), \text{ for,}
\]
\[
\begin{bmatrix} a & u \\ a^{-1}u^{-1}a & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -au^{-1} + u^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & u \\ \ast & \ast \end{bmatrix}
\]
To prove that \[
\begin{vmatrix}
 b \\
 a
\end{vmatrix}
\begin{vmatrix}
 b' \\
 a
\end{vmatrix} = \begin{vmatrix}
 bb' \\
 a
\end{vmatrix}
\] we have to check that :
\[
\begin{bmatrix}
 a & b \\
 a^{-1}c & d
\end{bmatrix}
\begin{bmatrix}
 a' & b' \\
 a^{-1}c' & d'
\end{bmatrix} = \begin{bmatrix}
 a & bb' \\
 * & *
\end{bmatrix}
\mod E(R)
\]

This follows from the following matrix identity :
\[
\begin{bmatrix}
 1 & 0 & 0 \\
 0 & 0 & 1 \\
 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
 a & b & 0 \\
 a^{-1}c & d & 0 \\
 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
 a & b' & 0 \\
 a'c' & d' & 0 \\
 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
 0 & 0 & -1 \\
 0 & 1 & 0 \\
 1 & 0 & a^{-1}b
\end{bmatrix}
\]
\[
\begin{bmatrix}
 a & bb' & 0 \\
 * & * & 0 \\
 * & * & 1
\end{bmatrix} = \begin{bmatrix}
 a & bb' & 0 \\
 * & * & 0 \\
 * & * & 1
\end{bmatrix}
\mod E(R)
\]

A combination of the above results yields :
\[
\begin{bmatrix}
 b \\
 a
\end{bmatrix} = \begin{bmatrix}
 b \\
 a+b
\end{bmatrix} = \begin{bmatrix}
 -a \\
 a+b
\end{bmatrix} = \begin{bmatrix}
 -b \\
 b
\end{bmatrix} = \begin{bmatrix}
 -1 \\
 b
\end{bmatrix} = \begin{bmatrix}
 a \\
 b
\end{bmatrix}
\]

and therefore finally :
\[
\begin{bmatrix}
 b \\
 a
\end{bmatrix} = \begin{bmatrix}
 b \\
 a'
\end{bmatrix} = \begin{bmatrix}
 bb' \\
 b
\end{bmatrix} = \begin{bmatrix}
 b_a \\
 b
\end{bmatrix} = \begin{bmatrix}
 b \\
 b
\end{bmatrix} = \begin{bmatrix}
 bb' \\
 b
\end{bmatrix}.
\]

**Lemma 4.5.** There exists exactly one bimultiplicative symbol
\[
\begin{vmatrix}
 b \\
 I
\end{vmatrix}
\]
\(
\in SK_1(R)
\)
which is defined whenever \( b \) and the nonzero ideal \( I \) are relatively prime, which depends only on the residue class of \( b \mod I \) and which coincides with the Mennicke symbol
\[
\begin{vmatrix}
 b \\
 a
\end{vmatrix}
\]
when \( I = Ra \).

**proof**
For any nonzero ideal \( I \) one can choose a relatively prime ideal \( J \) which is contained in the ideal class \( \langle I \rangle^{-1} \) (Lemma 2.5.), thus \( I.J = Rc \) for some \( c \). By the Chinese remainder theorem, there exists an element \( b' \) such that \( b' \equiv b \mod I \); \( b' \equiv 1 \mod J \). Now, suppose that there exists a symbol with the required properties, then
\[
\begin{vmatrix}
 b \\
 I
\end{vmatrix} = \begin{vmatrix}
 b \\
 I
\end{vmatrix} \begin{vmatrix}
 1 \\
 J
\end{vmatrix} = \begin{vmatrix}
 b' \\
 I
\end{vmatrix} \begin{vmatrix}
 b' \\
 J
\end{vmatrix} = \begin{vmatrix}
 b' \\
 c
\end{vmatrix},
\]
which proves uniqueness.

We have to prove that
\[
\begin{vmatrix}
 b \\
 I
\end{vmatrix} \overset{\text{def}}{=} \begin{vmatrix}
 b' \\
 c
\end{vmatrix}
\]
does not depend on the choices made.

The definition does not depend on the choice of \( b' \), for let \( b'' \) also
satisfy \( b'' = b \mod I, b'' = 1 \mod J, \) then \( b'' = b' \mod R_c, \) whence \( b'' \equiv b' \mod c \). 

Further, the definition does not depend on the choice of \( c \), for let \( uc \) be another generator for \( I.J \) (hence \( u \in U(R) \)), then \( \begin{vmatrix} b' \\ uc \\ c \end{vmatrix} = \begin{vmatrix} b' \\ u \\ c \end{vmatrix} = \begin{vmatrix} b' \\ c \end{vmatrix} \).

Finally, the definition does not depend on the choice of \( J \). For if \( J' \) is another ideal prime to \( I \) such that \( I.J' = R_c' \), then choosing \( K \) in the ideal class of \( I \) but prime to \( I \) we get: \( J.K = R_d, J'.K = R_d' \) for some elements \( d \) and \( d' \). Because \( cd' \) and \( c'd \) both generate the ideal \( I.J.J'.K \), \( cd'(c'd)^{-1} \in U(R) \). Finally, choose \( b' \) such that \( b'' \equiv b' \mod I, b'' \equiv 1 \mod J.J'.K, \) then \( b'' \equiv 1 \mod R_d, b'' \equiv 1 \mod R_d' \), whence:

\[
\begin{vmatrix} b'' \\ c \end{vmatrix} = \begin{vmatrix} b' \\ cd' \\ c'd \\ c' \end{vmatrix} = \begin{vmatrix} b'' \\ c \end{vmatrix}.
\]

We return to the proof of Th.4.1.: if \( R \) is an \( S \)-Dedekind domain and if \( P \) is a nonzero prime ideal of \( R \), then the correspondence \( b \to \begin{vmatrix} b \\ P \end{vmatrix} \) is a homomorphism from the group \( (R/P)^* \) to \( K_1(R) \). Therefore, this morphism factorizes through \( (R/P)^* \to K_1(R/P) \). Forming the direct sum over all nonzero primes, we obtain the required homomorphism \( \otimes K_1(R/P) \to K_1(R) \).

Let us proof the exactness of the sequence: \( \otimes K_1(R/P) \to K_1(R) \to K_1(F) \).

The image of the first homomorphism is clearly the subgroup of \( K_1(R) \) generated by all Mennicke symbols \( \begin{vmatrix} b \\ a \end{vmatrix} \), and hence it is equal to the image of \( SL_2(R) \) in \( K_1(R) \). The kernel of the second morphism is clearly \( SK_1(R) \) and by Prop. 4.2. these two groups are equal.

Before defining a morphism \( K_2(Q) \to \otimes K_1(R/P) \), we will recall some results due to S. Green (5) on the Milnor group of a skewfield.

For any \( x \in [Q^*, Q^*] \), let \( a_x \in SL_n(Q) \) be the matrix with diagonal entries \( (1, \ldots, 1, x, 1, \ldots, 1) \) with \( x \) in the \( i \)-th place. Define elements \( b_x \in St_n(Q) \) in the following way. Let \( b_x \in St_n(Q) \) be any cross-section for \( a_x \), \( b_1(1) \). For \( i \neq 1 \), let \( b_i(x) = w_{ij}(1)b_i(x)w_{ij}(1)^{-1} \) where \( w_{ij}(u) = \chi_{ij}^u \chi_{ji}^{-u}^{-1} \chi_{ij}^u \) for any unit \( u \) in \( Q \).
Therefore, if \( u, v \in Q^* \), there exist elements \( c_{ij}(u,v) \in K_2(n,R) = \ker(\text{St}_n(R) \rightarrow \text{SL}_n(R)) \) such that:

\[
c_{ij}(u,v)b_1([u,v]) = h_{ij}(u)h_{ij}(v)h_{ij}(vu)^{-1} \quad \text{where} \quad h_{ij}(u) = w_{ij}(u)w_{ij}^{-1}.
\]

It is easily checked that \( c_{ij}(u,v) \) is independent from \( i \) and \( j \), whence we will denote \( c(u,v) = c_{ij}(u,v) \).

For \( x,y \in [Q^*, Q^*] \) we define elements \( d(x,y) \in K_2(n,R) \) to be

\[
d(x,y) = b_1(x)b_1(y)b_1(xy)^{-1}.
\]

The main result is the following noncommutative version of Matsumoto's theorem:

**Theorem 4.6**: (S. Green) The abelian group \( K_2(R) \) has a presentation in terms of generators and relations as follows. The given generators \( c(u,v), d(x,y) \) with \( u,v \in Q^* \), and \( x,y \in [Q^*, Q^*] \) are subject only to the following relations and their consequences. (Let \( t,u,v \in Q^* \) and \( x,y,z \in [Q^*, Q^*] \))

(R1) \( d \) is a normalized 2-cocycle, i.e. \( d(x,y)d(xy,z) = d(y,z)d(x,yz) \) and

\[
d(1, z) = 1 = d(z, 1)
\]

(R2) \( c(tu,v) = c(v,t)^{-1}c(u,v)c(t, [u,v])d([t, [u,v]], [u,v])d([tu, v], [v, t])^{-1} \)

(R3) \( c(u, vz) = c(u,v)c(u, z)d(uzu^{-1}, [u, v]z^{-1})d([u, v], z^{-1})d(uzu^{-1}, z^{-1})^{-1} \)

(R4) \( c(u-v^{-1}, v) = c(u, v-u^{-1}) \)

Using this theorem, we state:

**Proposition 4.7**: (Tate, noncommutative) The group \( K_2(Q) \) is generated by those symbols \( c(u,v) \) and \( d(x,y) \) for which \( u \) and \( v \) are relatively prime elements of the \( S \)-Dedekind domain \( R \) and \( x, y \in [Q^*, Q^*] \).

**proof**

Let \( L \) be the subgroup of \( K_2(R) \) generated by all symbols \( d(x,y), c(u,v) \) with \( u \) and \( v \) relatively prime in \( R \). In view of Greens theorem we must prove
that $c(a,b) \in L$ for any $a,b \in \mathbb{Q}^*$. The proof will proceed by induction on the number of maximal ideals $P$ such that $v_P(a)v_P(b) \neq 0$.

Case 1: Suppose that there are no such prime ideals. In $F(R)$, $Ra = D^{-1}C$ for relatively prime ideals $C$ and $D$ in $R$ and similarly $Rb = F^{-1}E$, whence $C.D$ is relatively prime to $E.F$ by hypothesis.

Lemme 2.5 provides us with an ideal $G$ which belongs to the ideal class $\langle D \rangle^{-1}$ and which is relatively prime to $E.F$. Let $D.G = Rc$ and $d = c.a \in C.G$, then we have expressed $a$ as a quotient $c^{-1}.d$ with $c$ and $d$ relatively prime to $E.F$. Similarly, one can write $b$ as a quotient $e^{-1}.f$ with $e$ and $f$ prime to both $c$ and $d$. Now, using the relations (R2) and (R3) of Th.4.6, it is possible to express $c(a,b)$ as a product of elements of $L$ (using the fact that $[\mathbb{Q}^*,\mathbb{Q}^*] \subset U(R))$.

Case 2: Suppose that there is exactly one prime ideal $P$ of $R$ such that $v_P(a)v_P(b) \neq 0$, then we can choose an element $z \in \mathbb{Q}^*$ satisfying $v_P(z) = -1$, $v_P(z) \geq 0$ and $v_P(z)v_P(b) = 0$ for any other prime ideal $P'$ of $R$. Now, if $v_P(a) = 1$ and $v_P(b) = j$, then $c(z^ia,b) \in L$ by case 1 (because $v_{P'}(z^ia)v_{P'}(b) = 0$ for all $P' \in \text{Spec}(R)$). Therefore, if we are able to prove that $c(z,b) \in L$, then $c(a,b) \in L$ by (R2).

Clearly, $v_{P'}(z)v_{P'}(b(1-z)^j)) = 0$ for any $P' \in \text{Spec}(R)$. Therefore, $c(z,b(1-z)^j) \in L$ by case 1. It is rather easy to derive from the relations (R1) to (R4) that $c(z,1-z) = 1$, whence $c(z,b) \in L$.

Case 3: Suppose that there are $n$ distinct primes $P_1, \ldots, P_n$ for which $v_{P_1}(a)v_{P_1}(b) \neq 0$. Choose an element $z$ in $Q$ which satisfies:

$v_{P_1}(z) = -v_{P_1}(a)$ and $v_{P_1}(z)v_{P_1}(b) = 0$ for $P' \neq P_1$. By the induction hypothesis, both $c(az,b)$ and $c(z,b)$ belong to $L$ whence $c(a,b) \in L$.

As in the commutative case, the morphism $K_2(\mathbb{Q}) \rightarrow K_1(R/P)$ will be given by assigning continuous Steinberg symbols (so called 'tame' symbols) to the principal valuations defining $R$. 
Let us recall the following noncommutative definition of Steinberg symbols, which is due to S. Green (5):

**Definition 4.8.** A Steinberg symbol on a skewfield $Q$ consists of maps:

- $C: Q^* \times Q^* \to A$
- $D: [Q^*, Q^*] \times [Q^*, Q^*] \to A$

to an abelian group $A$ such that $C$ and $D$ satisfy analogue relations as $(R1)$ to $(R4)$.

Now, if $v$ is a principal valuation on $Q$, $Q_v$ the associated principal valuation ring and $M_v$ its maximal ideal, then we associate to $v$ the following Steinberg symbol:

- $C_v: Q^* \times Q^* \to (Q_v/M_v)^* \quad ab \quad C_v(u,v) = (-1)^v(x)v(y) x v(y)^{-v(x)}$
- $D_v: [Q^*, Q^*] \times [Q^*, Q^*] \to (Q_v/M_v)^* \quad ab \quad D_v(x,y) = 1$

It is left as an exercise to the reader to verify that $C_v$ and $D_v$ actually satisfy $(R1)$ to $(R4)$.

Define a homomorphism $D: K_2(Q) \to \oplus K_1(R/P)$ in the following way.

Each generator $c(u,v)$ is to map to the direct sum whose $p$-th component is the tame symbol $d_{v_P}(u,v)$, and each generator $d(x,y)$ is to map to the identity.

The next proposition completes the proof of Th.4.1.:

**Proposition 4.9.** The composite morphism $K_2(Q) \to \oplus K_1(R/P) \to K_1(R)$ is the identity.

**proof**

Using proposition 4.7, it is clearly sufficient to prove that the element $D(c(a,b))$ maps to the identity in $K_1(R)$ whenever $a$ and $b$ are relatively prime elements of $R$. 
Let \( Ra = P_1^{n_1} \ldots P_r^{n_r} \) and \( Rb = Q_1^{n_1} \ldots Q_s^{n_s} \), then the image of \( c(a,b) \) in \( K_1(R/Q_j) \) is equal to the class of \( a^{n_j} \) in \( (R/Q_j)^\ast \) ab.

Similarly, the image of \( c(a,b) \) in \( K_1(R/P_1) \) is the inverse of the class of \( b^{m_1} \) in \( (R/P_1)^\ast \) ab.

Thus, the image of \( D(c(a,b)) \) in \( K_1(R) \) is the quotient of:

\[
\begin{pmatrix}
| a_1 | & \ldots & | a_s | \\
| Q_1 | & \ldots & | Q_s |
\end{pmatrix}
= 
\begin{pmatrix}
| a_1 | & \ldots & | a_s | \\
| Q_1 | & \ldots & | Q_s |
\end{pmatrix}
= 
\begin{pmatrix}
| a | \\
| b |
\end{pmatrix}

by

\[
\begin{pmatrix}
| b_1 | & \ldots & | b_r | \\
| P_1 | & \ldots & | P_r |
\end{pmatrix}
= 
\begin{pmatrix}
| b_1 | & \ldots & | b_r | \\
| P_1 | & \ldots & | P_r |
\end{pmatrix}
= 
\begin{pmatrix}
| a | \\
| b |
\end{pmatrix}

and lemma 4.4. finishes the proof.

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