On a Problem of R.A. Hirschfeld.

D. Introduction.

One of the classical results in functional analysis is the Gelfand-Mazur theorem which states that every complex Banach divisor algebra is isomorphic to $\mathcal{C}$. More generally, I. Kaplansky [3] proved that every complex Von Neumann regular Banach algebra is finite dimensional over $\mathcal{C}$.

In recent years, R.A. Hirschfeld [2] and A. Verschoren [8] have extended Kaplansky's theorem in another direction.

R.A. Hirschfeld characterized all complex topological Von Neumann regular algebras which are finite dimensional over $\mathcal{C}$, and, on his suggestion, A. Verschoren extended his work to complex $\pi$-regular algebras. Verschoren obtained the following result.

**Theorem 1.** [8]

Let $A$ be a complex $\pi$-regular topological algebra satisfying the following conditions:

(a) $A$ is a Fréchet algebra (i.e. complete metric)
(b) $A/J(A)$ contains no strict field extension of $\mathcal{C}$
(c) $A$ contains no small idempotents

then $A/J(A)$ is finite dimensional over $\mathcal{C}$.

Here, $J(A)$ denotes the (ringtheoretical) Jacobson-radical of $A$, cfr. e.g. [5], and $A$ is said to contain no small idempotents if its origin has a neighbourhood containing no nontrivial idempotent element. E.g. the open unit ball of a complex Banach algebra does not contain any idempotent.

R.A. Hirschfeld asked whether one could extend Theorem 1 to complex topological algebras which are polynomial-regular, i.e. for every $a \in A$ there exists a polynomial $f(X) \in \mathcal{C}[X]$ with zero constant term and an
element \( b \in A \) such that \( f(a) \cdot b \cdot f(a) = f(a) \). The main purpose of this note is to answer this question affirmatively. A secondary aim is to show that the arguments given in [2] and [8] can be shortened considerably by using the classical Artin-Wedderburn structure theorem, cfr. e.g. [1].

1. **Polynomial regular Fréchet algebras.**

In this section we aim to prove the following result.

**Theorem 2.** Let \( A \) be a Fréchet algebra with identity which is polynomial regular and contains no small idempotents, then \( A/J(A) \) is semisimple Artinian.

Throughout we will assume that \( A \) is a complex polynomial regular algebra and by an idempotent we will always mean a nonzero idempotent.

If \( a \in A \), we will denote by \( P_a \) the set of all polynomials \( f(X) \in \mathbb{C}[X] \) with zero constant term and with a minimal number of non-zero coefficients such that \( f(a) \cdot b \cdot f(a) = f(a) \) for some \( b \in A \). E.g. if \( e \) is an idempotent element, then \( P_e \) consists of monomials. Further, \( H(A) = \{ a \in A \mid \forall x \in A, \forall f \in P_{ax} : f(ax) = 0 \} \).

**Lemma 1:** If \( A \) is polynomial regular, then \( J(A) = H(A) \).

**Proof.** By definition \( a \in J(A) \) implies that \( 1-ay \) is invertible on the left for every \( y \in A \). Now, take \( x \in A \), \( f \in P_{ax} \) and an element \( b \in A \) such that \( f(ax)b \cdot f(ax) = f(ax) \). Then \( 1-f(ax)b = 1-ax \cdot h(ax)c \) for some polynomial \( h(X) \in \mathbb{C}[X] \) and hence there exists an element \( Z \in A \) such that \( Z(1-f(ax)b) = 1 \). Finally, \( f(ax) = Z(1-f(ax)c) \cdot f(ax) = 0 \) and hence \( J(A) \subseteq H(A) \). Conversely, let \( a \in H(A) \) then for every \( x \in A \) and every \( f(X) \in P_{ax} \) we have \( f(ax) = 0 \). Thus, for some \( r \in \mathbb{N} \) and some polynomial \( h \) we have:

\[
(\star) \ (ax)^r(1-ax \cdot h(ax)) = 0
\]
Now, take $e = (ax)^r h(ax)^r h(ax)^r$ then we get using (**) and the fact that
ax and h(ax) commute:

$$e^2 = (ax)^{2r} h(ax)^{2r} = (ax)^{2r-1} h(ax)^{2r-1} = \cdots = (ax)^r h(ax)^r = e$$

Because $H(A)A C H(A)$, $e$ is an idempotent element of $H(A)$. Every $f \in F_e$ is monic yielding that $e = 0$. Therefore, $ax \cdot h(ax)$ is nilpotent and hence

$(1 - ax \cdot h(ax))$ is invertible, its inverse being $1 + ax \cdot h(ax) + (ax)^r h(ax)^r \cdots + (ax)^{r-1} h(ax)^{r-1}$. Finally, using (**) we obtain that $ax$ is nilpotent, whence $1 - ax$ is invertible, so $a \in J(A)$.

**Lemma 2.** Let $A$ be a polynomial regular Fréchet algebra which contains no small idempotents, then $A$ contains no infinite family of commuting idempotents.

**Proof.** Suppose there is an infinite family of commuting idempotents

$p_n \in A$. Using completeness of $A$ we can find positive real numbers $\alpha_n \in (0,1)$ such that $\|p_n\| < 2^{-n}$ for each $\beta \in [0,\alpha_n]$ $\| \| \|$ being the

$F$ norm. Let $\lambda_n = \alpha_n^2$ then the sequences $\sum \lambda_n p_n$ and $\sum \lambda_n^{1/2} p_n$ are

absolutely convergent. Let $a = \sum \lambda_n p_n \in A$ then there exists an element

$b \in A$ and a polynomial $f(X) \in \mathbb{C}[X]$ with zero constant term such that

$f(a) \cdot b \cdot f(a) = f(a)$. Now, $g(a) = \sum g(\lambda_n) p_n \cdot b$. $\sum g(\lambda_n) p_n = \sum g(\lambda_n) p_n$.

Multiplying both terms on both sides with $p_r$ yields $g(\lambda_r) p_r \cdot g(\lambda_r) p_r$

$= g(\lambda_r) p_r$ entailing that for every $n \in \mathbb{N}$:

$$g(\lambda_n)^{1/2} p_n b g(\lambda_n)^{1/2} p_n = p_n$$

Now, $c = \sum g(\lambda_n)^{1/2} p_n$ exists by the choice of the $(\lambda_n) \in \mathbb{N}$. Multiplication

being jointly continuous in a complete metric algebra we obtain c.b.c. =

$$\lim_{N \to \infty} \sum_{n=1}^{N} p_n = p_n$$. But $A$ contains no small idempotents, hence the right side
does not exist.
Proof of Theorem 2. Using the proof of lemma 1, every nonzero right ideal in \( A/J(A) \) contains a nonzero idempotent. Using a result of Kaplansky's, it will therefore be sufficient to show that \( A/J(A) \) contains no infinite number of orthogonal idempotents. Assume otherwise, then a countable subset of them can be lifted to a family of orthogonal idempotents of \( A \) by [5, VIII Prop. 4.] and the fact that \( J(A) \) is a nil ideal. But this contradicts Lemma 2.

2. Finite dimensionality of \( A/J(A) \).

We are now in a position to answer Hirschfeld's question:

**Proposition 1.** Let \( A \) be a complex polynomial regular algebra satisfying satisfying the following conditions

(a) \( A \) is a Fréchet algebra

(b) \( A/J(A) \) contains no strict field extension of \( \mathcal{F} \)

(c) \( A \) contains no small idempotents

then \( A/J(A) \) is finite dimensional over \( \mathcal{F} \)

**Proof.**

It follows from theorem 2 and the Artin-Weddenburn result that

\[
A/J(A) \cong M_{k_1}(\Delta_1) \oplus \cdots \oplus M_{k_n}(\Delta_n)
\]

where \( \Delta_i \) is a division algebra. Condition (b) implies that \( \Delta_i \cong \mathcal{F} \) for each \( i \), finishing the proof.

Of course, condition (b) is very restrictive. In fact we have the following result: which seems to have been overlooked in [5]:

**Proposition 2.** If \( A \) is complex polynomial regular Fréchet algebra without small idempotents, then the following statements are equivalent:

(a) : \( A/J(A) \) contains no strict field extension of \( \mathcal{F} \)

(b) : \( A \) is algebraic over \( \mathcal{F} \)
Proof.

(a) $\Rightarrow$ (b): By proposition 1 we know that:

$$A/J(A) \cong M_{k_1}(\mathcal{C}) \oplus \ldots \oplus M_{k_n}(\mathcal{C})$$

Take any element $\overline{a} = (a_1, \ldots, a_n) \in A/J(A)$, then $\overline{a}$ satisfies a polynomial $f(X) \in \mathcal{C}[X]$, namely:

$$f(X) = \prod_{i=1}^{n} g_i(a_i)$$

where $g_i(a_i)$ is the characteristic polynomial of $a_i$. So, for any $a \in A$ there exists a polynomial $f(X) \in \mathcal{C}[X]$ such that $f(a) \in J(A)$. It follows from the proof of lemma 1 that $J(A)$ is a nil ideal. Therefore, there exists a natural number $m$ such that $f(a)^m = 0$. Finally, $a$ satisfies $f(X)^m$, finishing the proof.

(b) $\Rightarrow$ (a): $A/J(A)$ is algebraic and $\mathcal{C}$ algebraically closed.

References.


