An Explicit Description of $T_{3,2}$

by

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An Explicit Description of $\mathbf{T}_{3,2}$. 

Let $F$ be a field of characteristic different from 2. Consider the polynomial ring 

$$P_{m,n} = F[X_{ij}(l) : 1 \leq i,j \leq n, i \leq l \leq m]$$

and the so called generic $n$ by $n$ matrices:

$$X_l = (X_{ij}(l))_{ij} \in M_n(P_{m,n})$$

The ring of $m$ generic $n$ by $n$ matrices, $G_{m,n}$, is the $F$-subalgebra of $M_n(P_{m,n})$ generated by $\{X_1, \ldots, X_m\}$. The trace ring of $m$ generic $n$ by $n$ matrices, $T_{m,n}$ is the $F$-subalgebra of $M_n(P_{m,n})$ generated by $G_{m,n}$ and the elements $Tr(Y)$ where $Y \in G_{m,n}$.

Herstein [1] and Formanek, Halpin and Lie [2] have given an explicit description of the trace ring of 2 generic 2 by 2-matrices. It turned out that $R_{2,2}$, the center of $T_{2,2}$, is the polynomial ring

$$R_{2,2} = F[Tr(X_1), Tr(X_2), D(X_1), D(X_2), Tr(X_1X_2)]$$

and $T_{2,2}$ is the free $R_{2,2}$-module of rank four with generators $\{1, X_1, X_2, X_1X_2\}$.

In [5] Small and Stafford proved that $T_{2,2}$ has finite global dimension. We give here a shorter proof of this result:

**PROPOSITION:** $\text{gldim}(T_{2,2}) = 5$

**PROOF:** It is sufficient to prove that for any maximal ideal $m$ in $R_{2,2}$, $\text{gldim}((T_{2,2})_m) \leq 5$. Consider first the case that $m$ contains $(X_1X_2 - X_2X_1)^2$. It is easy to verify that $X_1X_2 - X_2X_1$ is a normalizing element of $T_{2,2}$ and the quotient

$$T_{2,2}/T_{2,2}(X_1X_2 - X_2X_1) = F[\overline{X_1}, \overline{X_2}, Tr(X_1), Tr(X_2)]$$

because $\overline{D(X_i)} = Tr(X_i).\overline{X_i} - \overline{X_i}^2$ and $\overline{Tr(X_1X_2)} = 2\overline{X_1}.\overline{X_2} + \overline{Tr(X_1)} - \overline{Tr(X_1)}.\overline{X_2} - \overline{Tr(X_2)}.\overline{X_1}$. Therefore

$$\text{gldim}((T_{2,2})_m/(T_{2,2})_m(X_1X_2 - X_2X_1)) = 4$$
and by a standard argument $\operatorname{gldim}((T_{2,2})_m) = 5$. Now, let $m$ be a maximal ideal in $R_{2,2}$, not containing $(X_1X_2 - X_2X_1)^2$. Because $R_{2,2}(X_1X_2 - X_2X_1)^2$ is the Formanek center of $T_{2,2}, (T_{2,2})_m$ is an Azumaya algebra over the regular domain $(R_{2,2})_m$ whence

$$\operatorname{gldim}((T_{2,2})_m) = \operatorname{gldim}((R_{2,2})_m) = 5$$

finishing the proof.

In the remaining part of this note we will give an explicit description of the trace ring of 3 generic 2 by 2 matrices. The center of this ring, $R_{3,2}$, was described by Formanek [3] in a rather laborious way. Working with 2 by 2-matrices, one uses basically only two identities

(1) : $A^2 - \operatorname{Tr}(A)A + D(A) = 0$

(2) : $AB + BA = \operatorname{Tr}(A)B - \operatorname{Tr}(A)Tr(B) + Tr(A)B + Tr(B)A$

Consider the $F$-subalgebra $R$ of $\Delta_{3,2}$, the generic division algebra of 3 generic 2 by 2-matrices, generated by the elements

$$\{\operatorname{Tr}(X_1), \operatorname{Tr}(X_2), \operatorname{Tr}(X_3), D(X_1), D(X_2), D(X_3), \operatorname{Tr}(X_1X_2), \operatorname{Tr}(X_1X_3), \operatorname{Tr}(X_2X_3)\}$$

Using the identities (1) and (2), one verifies that the $F$-subalgebra of $\Delta_{3,2}$, $R\{X_1, X_2, X_3\}$ is a finite module over $R$ generated by the elements

$$\{*\} = \{1, X_1, X_2, X_3, X_1X_2, X_2X_3, X_1X_2X_3\}$$

Because $T_{3,2} \subset R\{X_1, X_2, X_3\} \subset \Delta_{3,2}$, we get that $K\dim(R) = \operatorname{trdeg}_F(Z(\Delta_{3,2})) = 9$ by [4]. Therefore, the generating elements of $R$ are algebraically independent i.e. $R$ is the polynomial ring

$$F[\operatorname{Tr}(X_1), \operatorname{Tr}(X_2), \operatorname{Tr}(X_3), D(X_1), D(X_2), D(X_3), \operatorname{Tr}(X_1X_2), \operatorname{Tr}(X_1X_3), \operatorname{Tr}(X_2X_3)]$$

Further, $T_{3,2} \subset R\{X_1, X_2, X_3\} \subset T_{3,2}$ and $\operatorname{Tr}(R\{X_1, X_2, X_3\}) \subset R\{X_1, X_2, X_3\}$. This entails that $T_{3,2} = R\{X_1, X_2, X_3\}$. Now, let $K$ be the field of fractions of $R$, then

$$\dim_K(\Delta_{3,2}) = \dim_K(K\{X_1, X_2, X_3\}) \leq 8$$
because $K\{X_1, X_2, X_3\}$ has generating set ($\ast$). Further $\dim_K(Z(\Delta_{3,2})) \geq 2$ because $\text{Tr}(X_1X_2X_3) \notin K$. For otherwise, because $\text{Tr}(X_1X_2X_3)$ is linear in each of the generic matrices, this would entail that

$$\text{Tr}(X_1X_2X_3) = \alpha \text{Tr}(X_1)\text{Tr}(X_2)\text{Tr}(X_3) + \beta (\text{Tr}(X_1)\text{Tr}(X_2X_3) + \text{Tr}(X_2) + \text{Tr}(X_1X_3) + \text{Tr}(X_3)\text{Tr}(X_1X_2))$$

and by specializing $X_1|\rightarrow\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $X_2|\rightarrow\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $X_3|\rightarrow\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ one obtains a contradiction. Combining this, we get that

$$8 \leq \dim_K(Z(\Delta_{3,2}))\dim_{Z(\Delta_{3,2})}(\Delta_{3,2}) = \dim_K(\Delta_{3,2}) \leq 8$$

i.e. the set ($\ast$) is linearly independent over $K$ or $R$.

Taking traces in the identity

$$(X_1X_2X_3)^2 - \text{Tr}(X_1X_2X_3)X_1X_2X_3 + D(X_1)D(X_2)D(X_3) = 0$$

and simplifying the first term we get that $\text{Tr}(X_1X_2X_3)$ satisfies the quadratic equation:

$$(\ast\ast) : X^2 - AX + B = 0$$

where

$$A = \text{Tr}(X_1)\text{Tr}(X_2X_3) + \text{Tr}(X_2)\text{Tr}(X_1X_3) + \text{Tr}(X_3)\text{Tr}(X_1X_2) - \text{Tr}(X_1)\text{Tr}(X_2)\text{Tr}(X_3)$$

$$B = D(X_1)\text{Tr}(X_2X_3)^2 + D(X_2)\text{Tr}(X_1X_3)^2 + D(X_3)\text{Tr}(X_1X_2)^2$$

$$- \text{Tr}(X_1)\text{Tr}(X_2)\text{Tr}(X_1X_2)D(X_3) - \text{Tr}(X_1)\text{Tr}(X_3)\text{Tr}(X_1X_2)D(X_2)$$

$$- \text{Tr}(X_2)\text{Tr}(X_3)\text{Tr}(X_2X_3)D(X_1)$$

$$+ \text{Tr}(X_1)^2D(X_2)D(X_3) + \text{Tr}(X_2)^2D(X_1)D(X_3) + \text{Tr}(X_3)^2D(X_1)D(X_2)$$

$$- 4D(X_1)D(X_2)D(X_3) + \text{Tr}(X_1X_2)\text{Tr}(X_1X_2)\text{Tr}(X_2X_3)$$

So we proved the

**THEOREM**: If $R$ is the polynomial ring:

$$F[\text{Tr}(X_1), \text{Tr}(X_2), \text{Tr}(X_3), D(X_1), D(X_2), D(X_3), T(X_1X_2), T(X_1X_3), T(X_2X_3)]$$
(1) \( R_{3,2} \) is the free \( R \)-module of rank 2 generated by 1 and \( T(X_1X_2X_3) \). \( T(X_1X_2X_3) \) satisfies the quadratic equation (**) over \( R \).

(2) \( T_{3,2} \) is the free \( R \)-module of round 8 generated by

\[ \{1, X_1, X_2, X_3, X_1X_2, X_1X_3, X_2X_3, X_1X_2X_3 \} \]

Being free over a polynomial subring of the center, \( T_{3,2} \) is a reflexive module over \( R_{3,2} \). Further, we claim that the localization of \( T_{3,2} \) at a central height one prime is an Azumaya algebra. For, such a prime \( p \) cannot contain simultaneously the elements \( (X_1X_2 - X_2X_1)^2, (X_1X_3 - X_3X_1)^2, (X_2X_3 - X_3X_2)^2 \) belonging to the Formanek center, whence \((T_{3,2})_p\) is a localization of an Azumaya algebra. This proves that \( T_{3,2} \) is a reflexive Azumaya algebra.

This implies that \( T_{3,2} \) is not a free module over \( R_{3,2} \), since this would entail that \( T_{3,2} \) is an Azumaya algebra and dividing out the commutator ideal one finds an epimorphic image of smaller p.i. degree.

From \( m \) generic 2 by 2 - matrices one would similarly like to consider the \( F \)-subalgebra \( R \) of \( \Delta_{m,2} \) generated by the elements \( Tr(X_1), D(X_i), Tr(X_iX_j) \). But from \( m \geq 4 \) \( R \) can never be a polynomial ring because the number of generators in \( 2m + \binom{m}{2} \) whereas the \( K \text{dim}(R) = 4m - 3 \). Therefore, a similar approach fails for \( m \geq 4 \).

The description of \( R_{3,2} \) and \( T_{3,2} \) can also be applied to determine the Poincaré series. Clearly,

\[ P(R, t) = \frac{1}{(1 - t)^3(1 - t^2)^6} \]

since \( \text{deg}(Tr(X_i)) = 1 \) and \( \text{deg}(Tr(X_iX_j)) = \text{deg}(D(X_i)) = 2 \).

Therefore,

\[ P(R_{3,2}, t) = \frac{1 + t^3}{(1 - t)^3(1 - t)^6} \]

whence \( R_{3,2} \) cannot have finite global dimension, for otherwise the Poincaré series should have the form \( \frac{1}{f(t)} \) for some \( f(t) \in \mathbb{Z}[t] \). Finally,

\[ P(T_{3,2}, t) = \frac{1 + 3t + 3t^2 + t^3}{(1 - t)^3(1 - t^2)^6} = \frac{1}{(1 - t)^6(1 - t^2)^3} \]
REFERENCES.


