A NOTE ON CLIFFORD REPRESENTATIONS

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0. Introduction

That twisted group rings provide a link between the theory of Clifford algebras and the theory of projective representations of finite groups is well-known, even if perhaps it was more of a gadget to physicists, cfr. [Ra], than a tool to mathematicians. In the first section we establish that a projective representation of $B_n$, where $B_n$ is the $n$-fold product of $\mathbb{Z}/2\mathbb{Z}$, is determined by an $n$-dimensional quadratic form together with some well described subgroup of $B_n$. Any twisted group ring of $B_n$ is then a Clifford algebra over its center but we pay particular attention to the so called "Clifford representations" which yield Clifford algebras over the ground ring. In a short second section we introduce the notion of a Clifford group and provide some examples of these. We do not obtain a complete description of Clifford groups but we hope to create some interest in such a complete determination which may be the topic of some further investigations.

1. Clifford representations

In this section $R$ is a commutative ring such that 2 is a unit in $R$ and $B_n$, where $n \in \mathbb{N}$ stands for the abelian group $\mathbb{Z}/2\mathbb{Z} \times \ldots \times \mathbb{Z}/2\mathbb{Z}$, where $n$ copies of $\mathbb{Z}/2\mathbb{Z}$ appear. A projective representation of a group $G$ over $k$ is a group morphism

$$\phi : G \rightarrow PGL_n(k)$$

where $k$ is a field. Such a projective representation over a field $k$ may also be determined by a ring homomorphism

$$\phi^c : kG_c \rightarrow M_n(k)$$

where $c \in H^2(G, k^*)$ and $kG_c$ is the twisted group ring with respect to the factor system $c(\sigma, \tau)$ where $\sigma, \tau \in G$. First we generalize these concepts to the case where $k$ is merely a ring. If $P$ is a finitely generated projective $R$-module then $\text{End}_R(P)$ is an Azumaya algebra over $R$ and its group of units $\text{Aut}_R(P)$ contains the trivial units $U(R)$ in its center because such a unit defines an
automorphism of $P$. A **projective representation of $G$ over $R$** defined by $P$ is a group morphism $\pi : G \to \text{Aut}_R(P)/U(R)$. This may be interpreted as a map, again written $\pi, \pi : G \to \text{Aut}_R(P)$, satisfying the relations: $\pi(\sigma) \cdot \pi(\tau) = \pi(\sigma \cdot \tau)$ for all $\sigma, \tau \in G$, where $\pi(\sigma, \tau)$ is a factor set representing $c \in H^2(G, U(R))$ obtained by taking a transversal for $\pi(G)$ in $\text{Aut}_R(P)$. The twisted group ring $RG_c$ is the free $R$-module $\bigoplus_{\sigma \in G} Ru_\sigma$ with multiplication induced by the rules

$$u_\sigma \cdot u_\tau = c(\sigma, \tau)u_{\sigma \cdot \tau}$$

for all $\sigma, \tau \in G$. The $R$-bilinear extension of the map $G \to \text{Aut}_R(P)$ determines a ring homomorphism $\pi^c : RG_c \to \text{End}_R(P)$. For a given factor system $\{c(\sigma, \tau) : \sigma, \tau \in G\}$ we define $G^c = \{\sigma \in G : \forall \tau \in C_G(\sigma) : c(\sigma, \tau) = c(\tau, \sigma)\}$ where $C_G(\sigma)$ is the centralizer of $\sigma$ in $G$.

Now, consider $G = B_n$. Since $B_n$ is Abelian it is clear that the subring $(RB_n)_c$ coincides with the center of $(RB_n)_c$ (this follows for example immediately from the fact that the center of $(RB_n)_c$ is $B_n$-graded).

**Definition 1.1**: A projective representation of $B_n$ determined by a factor system $\{c(\sigma, \tau) : \sigma, \tau \in G\}$ is a Clifford representation if $B^c_n$ is minimal, i.e. $B^c_n = 0$ if $n$ is even and $B^c_n \simeq \mathbb{Z}/2\mathbb{Z}$ if $n$ is odd.

In order to see that this definition makes sense we should justify the discrepancy of the definition in the "$n$ is odd" case. Assume that $R = k$ is a field, then $(kB_n)_c$ is a semi-simple $k$-algebra, i.e. $(kB_n)_c = A_1 \oplus \cdots \oplus A_k$ for simple $k$-algebras $A_i$. If $i \neq 1$ then the center of $(kB_n)_c$ is strictly larger than $k$. If $k = 1$, then the dimension of the center is also strictly greater than one since $A_1$ is of square dimension over its center. So, in both cases we arrive at $B^c_n \neq 0$, what explains that for odd $n$ the minimal possibility for $B^c_n$ is to be a copy of $\mathbb{Z}/2\mathbb{Z}$.

Let us recall the definition of Clifford algebras. Let $S$ be any commutative ring, $V$ a free $S$-module of rank $n$ equipped with a nonsingular quadratic form $Q = \alpha_1 X_1^2 + \cdots + \alpha_n X_n^2$, where $\alpha_i \in U(S)$. The Clifford algebra $C(V, Q)$ is defined to be the quotient of the tensor algebra $T(V)$ with respect to the two-sided ideal generated by the elements $v \otimes v - Q(v)$ where $v \in V$. More specifically,

$$C(V, Q) = \bigoplus_{e_i \in \{0, 1\}} R.e_1^{e_1} \cdots e_n^{e_n}$$

with multiplication defined by the rules $e_i^2 = \alpha_i \in U(S)$ and $e_i.e_j = -e_j.e_i$ for $i \neq j$. The following theorem explains why a representation with minimal $B^c_n$ is called a Clifford representation.

**Theorem 1.2**: If the factor system $\{c(\sigma, \tau) : \sigma, \tau \in G\}$ where $G = B_n$, determines a Clifford representation, then $RG_c$ is isomorphic to a Clifford
algebra of a nonsingular quadratic form over $R$.

**Proof:**

Since $RG_e \simeq RG_{e'}$ if the factor systems $e$ and $e'$ are equivalent, i.e. when $e = e' \in H^2(G, U(R))$ we may assume that $e(\sigma, r)$ is normalized so that $e(\sigma, 0) = e(0, \sigma) = 1$ for all $\sigma \in B_n$. The first part of the proof consists in describing the cohomology group $H^2(B_n, U(R))$. We claim that

\[(*) \quad H^2(B_n, U(R)) \simeq \prod_{i=1}^n H^2(\mathbb{Z}/2\mathbb{Z}, U(R)) \times \prod_{i<j} M_{ij}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, U(R))\]

where each $i - j$-mixterm $M_{ij}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, U(R))$ equals $H^2_{\mu}(\mathbb{Z}/2\mathbb{Z}, U(R))$ which is the group of equivalence classes of 2-cocycles satisfying the relations:

$$f(\sigma_1 \sigma_2, r) = f(\sigma_1, r)f(\sigma_2, r); f(\sigma, r_1 r_2) = f(\sigma, r_1)f(\sigma, r_2)$$

The isomorphism $(*)$ may be easily verified by (a) : restricting the 2-cocycles to the cyclic components $\mathbb{Z}/2\mathbb{Z}$ of $B_n$ and (b) : associating to basis elements $e_i$ and $e_j$ an element $f$ of $M_{ij}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, U(R))$:

$$f : \mathbb{Z}/2\mathbb{Z}.e_i \times \mathbb{Z}/2\mathbb{Z}.e_j \to U(R); (\sigma, r) \to c(\sigma, r)r c(r, \sigma)^{-1}$$

Furthermore, we have that $H^2(\mathbb{Z}/2\mathbb{Z}, U(R)) \simeq U(R)/U(R)^2$, the isomorphism being given by the correspondence $f \to f(1,1)$, and also $H^2_{\mu}(\mathbb{Z}/2\mathbb{Z}, U(R)) = \mu_2 = \{+1, -1\}$. So, we finally obtain:

$$H^2(B_n, U(R)) = \prod_{i=1}^n U(R)/U(R)^2 \times \prod_{i<j} \mu_2$$

The foregoing states that each factor system corresponds (up to equivalence) in an unambiguous way to a quadratic form, viewed as an element of $\prod_{i=1}^n U(R)/U(R)^2$, together with some element $\zeta$ in $\prod_{i<j} \mu_2$. There are Clifford representations such that $\zeta$ is not equal to $\prod_{i<j}(-1)$. For example, consider $B_3$ as a $\mathbb{Z}/2\mathbb{Z}$-vectorspace with basis-elements $e_1, e_2, e_3$. We may determine a 2-cocycle $e(\sigma, r)$ by the following table

<table>
<thead>
<tr>
<th>$c(e_i, e_j)$</th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$e_2$</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$e_3$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
It is clear that $B^c_0 = \mathbb{Z}/2\mathbb{Z}.e_3$ and hence the given cocycle describes a Clifford representation. On the other hand, it is possible to change the $\mathbb{Z}/2\mathbb{Z}$-basis for $B_n$ as follows $(e_1, e_2, e_3) \rightarrow (e_1, e_2, e_1e_2e_3)$ then the new table for $c$ is the following

<table>
<thead>
<tr>
<th></th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_1e_2e_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$e_2$</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$e_1e_2e_3$</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

The corresponding element in $\prod_{i<j} \mathbb{R}_2$ is $(-1, -1, -1)$. In the next lemma we will prove that this is possible in general.

**Lemma 1.3**: Consider $f \in \prod_{i<j} \mathbb{R}_2$ such that the twisted groupring $(RB_n)_f = \bigoplus R.e_i^1 \cdots e_i^n$ with $e_i^2 = 1$, $e_ie_j = f_{ij}e_i e_i$ (i.e. the cocycle corresponding to the quadratic form $(1, \cdots, 1)$ and the element $f$ given above), is a Clifford representation. Then one can change the basis of $B_n$ such that in this new basis the element in $\prod_{i<j} \mathbb{R}_2$ corresponding to it is $(-1, \cdots, -1)$.

**Proof**:

The proof is by induction on $n$, since there is nothing to prove for $n = 1, 2$. First consider the case where $n$ is odd and $n \geq 3$. By our assumption, $B^f_n \cong \mathbb{Z}/2\mathbb{Z}$. Let $\sigma$ be a generator of this subgroup. We have that $B_n/B^f_n \cong B_{n-1}$ and let $H \cong B_{n-1}$ be the complement for $<\sigma>$ in $B_n$ and let $g$ be the factor system induced by $f$ on $H$. It is clear that $H^g = 0$ and therefore $g$ determines a Clifford representation of $H \cong B_{n-1}$. By induction we may assume that a basis for $H$, say $d_1, \cdots, d_{n-1}$, has been chosen such that the corresponding element in $\prod_{i<j}^{n-1} \mathbb{R}_2$ equals $(-1, \cdots, -1)$. Now, we may construct a basis for $B_n$ by taking $\{d_1, \cdots, d_{n-1}, d\}$ where $d = \sigma d_1 \cdots d_{n-1}$. It is easily verified that the element of $\prod_{i<j} \mathbb{R}_2$ corresponding to the selected basis is exactly equal to $(-1, \cdots, -1)$ as claimed. Next, we consider the case where $n$ is even. Let $H \cong B_{n-1}$ be the complement of $<e_n>$ in $B_n$ and let $g$ be the induced factor system of $f$ on $H$. Let us check that $g$ determines a Clifford representation of $B_{n-1}$. First, note that $H^g \neq 0$ because $n - 1$ is odd. On the other hand, $H^g$ cannot contain some $\mathbb{Z}/2\mathbb{Z}.\sigma \oplus \mathbb{Z}/2\mathbb{Z}.\tau$ with $\sigma, \tau \in H$ because, in case $f(\sigma, e_n) = 1$ or $f(\tau, e_n) = 1$ then $B^f_n \neq 0$, a contradiction, while in the other case $f(\sigma, e_n) = f(\tau, e_n) = -1$ and we find that $\sigma \tau \in B^f_n$, again a contradiction. Applying the induction hypothesis we may choose a basis $c_1, \cdots, c_{n-1}$ of $H$ such that the corresponding element in $\prod_{i<j}^{n-1} \mathbb{R}_2$ is $(-1, \cdots, -1)$. Also by induction and the first part of the proof, we may assume that the central element in $(RH)_g$ equals $c_1 \cdots c_{n-1} = \sigma$. Since $H^f = 0$ we have $\sigma e_n = -e_n$, and so $e_n$ anti-commutes with an odd number of the $c_i$ say $c_{i_1}, \cdots, c_{i_k}$. Take as the new $n$-th basis vector for $B_n$ the
element \( \omega = e_ne_i \cdots e_{i_n} \). It is easily checked that \( \omega e_j = -e_j \omega \) for all \( j \leq n - 1 \) and this finishes the proof of the lemma.

Now, this also finishes the proof of the theorem because, in the new basis we clearly see that the relations \( e_i^2 = \alpha_i \in U(R) \) and \( e_i e_j = -e_j e_i \) for \( i \neq j \) do hold indeed.

**Remark 1.4**: For an arbitrary projective representation of \( B_n \) determined by a factor system \( \{ c(\sigma, r) : \sigma, r \in B_n \} \) it is always true that \( (RB_n)_c \) is a Clifford algebra over the ring \( (RB_n)_c \) (i.e. over its center which is determined by the ray classes with respect to \( c \)).

In the proof of theorem 1.2 we have established that a Clifford representation of \( B_n \) say \( (RB_n)_c \) is isomorphic to the Clifford algebra associated to the quadratic form \( < c(e_1, e_1), \cdots, c(e_n, e_n) > \) (in diagonal form) where \( e_1, \cdots, e_n \) is a suitable \( \mathbb{Z}/2\mathbb{Z} \)-basis for \( B_n \). It is now easy to study the splitting problem for a Clifford representation. Let \( S \) be the free extension of \( R \) obtained by adjoining \( \sqrt{c(e_i, e_i)} \) for \( 1 \leq i \leq n \), then we obtain \( (SB_n)_c \cong C(S^{n}, < 1, \cdots, 1 >) \cong M_{2^n}(S) \) when \( n \) is even and \( \bar{m} = \frac{n}{2} \) or \( (SB_n)_c \cong M_{2^n}(S) \oplus M_{2^n}(S) \) when \( \bar{n} \) is odd and \( t = \frac{n-1}{2} \).

It is easy to verify that \( (SB_n)_c \) is an epimorphic image of the groupring \( SG_n \) where \( G_n \) is a finite central extension of \( B_n \). In the case considered here we can give a complete description of such a group. Put \( G_n = < a_1, \cdots, a_n, b > \) with \( a_i = 1 \) and \( [a_i, a_j] = b \) for all \( i \neq j \). An immediate consequence is that \( b^2 = 1 \) and that \( b \) is central in \( G_n \). We may now define a ringepimorphism \( \phi : SG_n \rightarrow (SB_n)_c \) by sending \( b \) to \(-1\) and \( a_i \) to \( \frac{1}{\sqrt{c(e_i, e_i)}} e_i \). So, we may view the (projective) Clifford representations of \( B_n \) in a generic way as common representations of \( G_n \) up to splitting the representation by passing to an extension \( S \) of \( R \) in the well-described way above.

2. Clifford groups; an introduction

In classical representation theory, projective representations appear in Clifford's theorem describing a representation of a group in terms of a projective representation of some subgroup(s). After the foregoing section, the natural question is to investigate then has to be: which groups \( G \) have the property that the groupring \( RG \) splits into Clifford representations over the respective c\( \ddot{e} \)ntr\( \ddot{a} \), i.e.

\[
RG = \bigoplus_{i=1}^{\bar{n}} (R \cdot B_{n_i})_{e_i}
\]

in particular, when \( RG \) split in Clifford representations over \( R \), i.e.
Groups with the first property are called **general Clifford groups**, groups with the second property are called **Clifford groups**. Since every abelian group is a general Clifford group the second concept is much more restrictive; for general Clifford groups $G$ one should be more interested in $G/Z(G)$.

In the sequel we will always assume that $| G |^{-1} \in R$.

**Proposition 2.1** : If $G$ is a (general) Clifford group and $H$ is a normal subgroup of $G$, then $G/H$ is a (general) Clifford group.

**Proof** : Let $\omega_H$ be the augmentation ideal of $H$ in $RG$. Since $| G |$ is a unit in $R$, $RG$ is an Azumaya algebra and hence the canonical epimorphism

$$\pi_H : RG \rightarrow R(G/H)$$

maps the center onto the center. If $RG$ decomposes as $A_1 \oplus \cdots A_n$, where each $A_i$ is a Clifford algebra (over its center), then a similar result holds for $R(G/H) = \overline{A_1} \oplus \cdots \oplus \overline{A_n}$ (note that each $A_i$ is Azumaya, even if it is possible that 2 does not divide $| G |$).

**Proposition 2.2** : Any group $G$ such that the commutator subgroup $G'$ is central and such that $G/G' \simeq B_n$ for some $n$ is a general Clifford group.

**PROOF** : Since $RG'$ is central in $RG$ we may put $RG' = R'$ and view $RG$ as $R'(G/G')_c$, where $c$ is some 2-cocycle obtained by a selection of a transversal of $G'$ in $G$. By remark 1.4, $R'(G/G')_c \simeq (R'B_n)_c$ is a Clifford algebra over its center, hence the result follows.

**Example 2.3** : The groups $G_n$ constructed at the end of the foregoing section are Clifford groups.

Clearly, the more difficult and as yet unsolved problem is to describe the Clifford groups more completely. This problem may not be very easy, for example the fact that the product of Clifford groups need not be a Clifford group (verify for $D_4 \times D_4$ where $D_4$ is the dihaeder group of order 8) is already an obstruction.

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