Center of Generic Division Algebras
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Abstract.
We link the study of the rationality problem of the center of the generic division algebra to geometrical moduli problems. Further, we compute the zeta functions up to dimension 8 and give a short proof of retract rationality in the prime case.

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1. Introduction

In 1972, S. Amitsur startled the world by showing that division algebras are not necessarily crossed products [1]. Crucial to his proof were the generic division algebras $\Delta_{m,n}$ which are defined as follows. Let $K$ be any field, then the ring of $m$ generic $n$ by $n$ matrices $\mathbb{G}_{m,n}$ is the subalgebra of $M_n(K)$ generated by the generic matrices $X_k = (x_{ij}(k))_{i,j}$. $\mathbb{G}_{m,n}$ is a domain and hence has a division ring of fractions which is $\Delta_{m,n}$. One of the main open problems is

Question: Determine all $(m,n)$ such that $K_{m,n}$, the center of $\Delta_{m,n}$, is (stably) rational over $K$.

Procesi [19] proved that $K_{m,n}$ is rational whenever $K_{2,n}$ is, thereby reducing the problem to 2 generic matrices. He also established rationality of $K_{2,2}$. Later Formanek [7],[8] proved rationality of $K_{2,3}$ and $K_{2,4}$.

There are some results supporting the rationality conjecture: Saltman [21] proved that the Brauer group of a smooth model of $K_{m,n}$ is reduced to $Br(K)$ (as should be the case for stably rational fields by some results of Grothendieck). He also showed that $K_{2,p}$ is retract rational whenever $p$ is a prime number. Colliot-Thélène and Sansuc [4] gave an improvement of this result. On the other hand, there is a cellular decomposition conjecture in the representation theory of finite dimensional hereditary algebras due to V. Kac [9] which would immediately imply rationality of $K_{2,n}$. Evidence supporting this conjecture can be gathered by calcu-
lating the zeta-functions of the parametrizing varieties. In [13] these computations were shown to be consistent with the conjecture up to 5 by 5 matrices.

The aim of this short note is threefold. First we want to recall some recent results relating $K_{2,n}$ to some moduli-problems in algebraic geometry. One can safely argue that the rationality of the center of the generic division algebra is the main open rationality problem. We give here an application of this geometrical connection, namely that $K_{m,n}$ is the field of rational matrix-invariants over an algebraically closed field in any characteristic. Perhaps the nonzero characteristic case of this result is new, Procesi proved the result in characteristic zero [20]. Next we extend the zeta-function calculations up to dimension 8 because $K_{2,8}$ is assumed to be the first testcase for a counterexample to the rationality conjecture. All results turn out to be compatible with Kac’s conjecture. Finally, we sketch a systematic approach to the problem based on the theory of tori-invariants. The strategy is implicit in Formanek’s proof in the 4 by 4 case. We give an illustrative short proof of Saltman’s retract rationality result.

2. Moduli spaces

In characteristic zero Procesi proved that $K_{2,n}$ is the field of rational matrix-invariants. That is, $GL_n(\mathbb{F})$ acts on $M_n(\mathbb{F}) \oplus M_n(\mathbb{F})$ by simultaneous conjugation and $K_{2,n}$ coincides with the corresponding fixed field. This result is crucial in proving the following connections between $K_{2,n}$ and moduli problems in algebraic geometry

(a) : moduli spaces of vector bundles over $\mathbb{P}^2$

A very coarse classification of all vector bundles over $\mathbb{P}^2$ is given by their topological invariants such as the rank and the Chern numbers. Given such parameters $r, c_1$ and $c_2$ one wants to study sufficiently general bundles with these invariants. They turn out to be stable which means that for all coherent subsheaves $\mathcal{F}$ of our bundle $\mathcal{E}$ we have that $c_1(\mathcal{F})/rk(\mathcal{F}) < c_2/r$. One can then construct a variety $M(r, c_1, c_2)$ whose points correspond to isoclasses of stable bundles of rank $r$ and Chern-numbers $c_i$. This variety is called the moduli space of stable rank $r$ bundles having Chern-numbers $c_1$ and $c_2$. In [12] it was proved that

**Theorem 1 :** $K_{2,n}$ is the function field of the moduli space $M(n,0,n)$
Modulo the rationality results of Formanek mentioned before this result implies rationality of $M(n,0,n)$ for $n \leq 4$. So far, only the case $n = 2$ was known.

(b) : Halfcanonical divisors on plane curves

Recall that the projective space $\mathbb{P}^{\frac{1}{2}n(n+3)}$ parametrizes plane projective curves of degree $n$. Let $U$ be an open subvariety consisting of nonsingular curves. Consider the flagvariety $W \subset \mathbb{P}^2 \times U$, then the natural projection $W \rightarrow U$ is a flat bundle of smooth curves.

We can then investigate the relative Picard scheme introduced by Grothendieck and studied by Artin [2] and Mumford [18]. This functor $\text{Pic}_{W/U}$ associates to a $U$-scheme $S$ the quotient group of the group of all invertible sheaves on $W \times_U S$ by the subgroup of those of the form $p_2^*(K)$ for $K$ on $S$ where $p_2$ is the projection on the second factor. With $\text{Pic}_{W/U}^d$ one denotes the subsheaf consisting of invertible sheaves of degree $d$. The sheafification of this functor with respect to the flat topology is represented by the variety $\text{Pic}_{W/U}^d$ whose points are couples $(C, L)$ where $C$ is a nonsingular plane curve of degree $n$ and $L$ a divisor on $C$ of degree $d$. With this terminology we have the following beautiful result.

**Theorem 2 (Van den Bergh [22]) :** $K_{2,n}$ is the function field of the relative Picard scheme $\text{Pic}_{W/U}^d$ for $d = \frac{1}{2}n(n - 1)$

Note that since the degree of the canonical divisor of a smooth curve of degree $n$ has degree $n(n - 1)$ this variety can be viewed as the generic variety of halfcanonical divisors of plane curves.

(c) : the $r$ subspace problem

The $r$-subspace asks for the classification of $r$ subspaces in a finite dimensional vectorspace up to basechange. Let us denote the dimension of the big space by $a_0$ and those of the subspaces by $a_i$, then the geometrical problem is that of studying $GL(a_0)$-orbits in the variety

$$\text{Grass}(a_1, a_0) \times \ldots \times \text{Grass}(a_r, a_0)$$

which was one of the testing examples for Mumford's [18]. It turns out that one can only have a nice quotient variety provided there are stable points for which Mumford gave a combinatorial criterion. When one has stable points one can form
the quotient variety. Apart from some easy cases nothing seems to be known about the rationality of these varieties. Implicit in the work of the first author and A. Schofield [14] is the following

**Theorem 3:** Suppose there are stable points for the natural action of $GL(a_0)$ on $Grass(a_1, a_0) \times ... \times Grass(a_r, a_0)$. If $n = \gcd(a_0, a_1, ..., a_r)$ then $K_{2,n}$ is stably equivalent to the functionfield of the quotient variety for this action.

Again, Formanek's results do imply an abundancy of stable rational settings for the $r$-subspace problem for which there does not seem to exist a geometric proof.

It is rather amusing that Formanek's rationality results give new geometrical results. However, there are virtually no geometrical results known which would give a new ringtheoretical result about $K_{2,n}$ (disregarding false results such as in [17]). One of the rare exceptions is the following:

**Theorem 4:** The center of the generic division algebra coincides with the field of rational matrixinvariants over an algebraically closed field (disregarding the characteristic).

**Sketch of proof:** The field of rational matrixinvariants is always the functionfield of the moduli space $M(n, 0, n)$, see e.g. [16] modulo the translation as in [12]. Moreover, in [16, section 5] Maruyama shows that this functionfield is always the field of invariants of a rational field under action of the symmetric group [16, Th. 5.8]. Restricting to the case of interest to us (i.e. rank $n$) and using the translation to generic matrices, one recovers the description due to Procesi and Formanek [7] of the center of the generic division algebra (over any field) as the $S_n$-invariants of a rational field (see also the last section).

### 3. Zeta functions

In this section we aim to calculate the zeta-functions of the varieties parametrizing indecomposable couples of matrices under simultaneous conjugation, up to dimension 8 extending the previous computations [13]. These calculations can be viewed as further evidence for the following conjecture due to V. Kac: the varieties parametrizing isoclasses of representations of a given dimension vector of any quiver
admit a cellular decomposition into a finite union of locally closed subvarieties each of which is some affine space. Clearly, a positive solution to this conjecture in the special case of matrix invariants would imply rationality of $K_{m,n}$. Unfortunately, there is not much evidence supporting this conjecture. For matrix invariants only the cellular decomposition of two 2 by 2 matrices is explicitly known [13]. We will now describe the implications of this conjecture on the zeta-functions and how one can calculate them.

With $Ind(n) \subseteq M_n(\mathbb{F}) \oplus M_n(\mathbb{F})$ we denote the set of indecomposable representations i.e. those which cannot be brought into proper block form by simultaneous conjugation. It is well known that $I(n)$ is a constructible subvariety. With $I(n)$ we denote the variety parametrizing the $GL_n(\mathbb{F})$-orbits on $Ind(n)$, for details we refer to [10]. We want to compute the zeta-function of this variety by counting its number of points over a finite field $\mathbb{F}_q$.

Hence, let $A(n)(\mathbb{F}_q)$ be the set of absolutely indecomposable representations of $\mathbb{F}_q$ up to simultaneous conjugation. If Kac's conjecture were true one would have

$$a_n(q) = \#A(n)(\mathbb{F}_q) = a_N q^N + \ldots + a_0$$

with $a_i$ the number of cells of dimension $i$ in the conjectural cellular decomposition. In particular they all have to be positive natural integers. Define the zeta-function to be

$$\zeta_{n,\mathbb{F}}(z) = \exp \left( \sum_{k=1}^{n} \frac{1}{k} a_n(q^k) z^k \right)$$

then this functions would have to have the following rational form

$$\zeta_{n,\mathbb{F}}(z) = (1 - q^N z)^{-a_N} \ldots (1 - qz)^{-a_1} (1 - z)^{-a_0}$$

These functions can be computed as follows: let $o_n(q)$ be the total number of orbits of $GL_n(\mathbb{F}_q)$ acting on $M_n(\mathbb{F}_q) \oplus M_n(\mathbb{F}_q)$ by conjugation and $i_n(q)$ (resp $a_n(q)$) will be the number of points in $I(n)(\mathbb{F}_q)$ and $A(n)(\mathbb{F}_q)$. Let $\Phi$ be the set of irreducible polynomials in $\mathbb{F}_q[t] - \{t\}$ and look at all functions from $\Phi$ to the set of all partitions of nonnegative integers $Par$ with finite support

$$\pi : \{f_1, \ldots, f_{k(\pi)}\} \to Par$$

such that $|\pi| = \sum \deg(f_i)$, $|\pi(f_i)| = n$. Let $\text{irr}(\pi)$ denote the number of ways one can assign an ordered set of $k(\pi)$ polynomials such that the $i$-th polynomial has
degree $v_i$, then one can show, see [Ka], that

$$o_n(q) = \sum_{\pi} \frac{\text{irr}(\pi) \prod_{l=1}^{k(\pi)} v_l \prod_{j=1}^{k_i} \pi_j(f_i)^2}{\prod_{l=1}^{k(\pi)} \prod_{j=1}^{m(l,\pi)} (1 - q^{-j v_l})}$$

where $\pi(f_i)'$ denotes the associated partition and $m(i, l)$ the multiplicity of the number $l$ in $\pi(f_i)$. In order to deduce from this $i_n(q)$ one simply uses Krull-Schmidt and previously determined $i_k(q)$ for $k < n$. Finally, the numbers $a_n(q)$ (and hence the zeta-function) is determined using the recursion formula

$$i_n(q) = \sum_{d|n} \frac{1}{d} \sum_{e|d} \mu(e) a_{n/d}(q^{d/e})$$

Thus, we have reduced everything to calculable data and we find

**Theorem 5**: The zetafunctions $\zeta_{n,q}(z)$ have a rational form consistent with the Kac-conjecture for all $n \leq 8$

Let us give the obtained polynomials:

$$a_1(q) = q^2, a_2(q) = q^5 + q^3, a_3(q) = q^{10} + q^8 + q^7 + q^6 + q^5 + q^4$$

and writing only the coefficients which run from $q^{n^2+1}$ downto $q^{n+1}$ we have

$$a_4(q) = (1, 0, 1, 1, 2, 1, 3, 2, 4, 2, 3, 1, 1)$$

$$a_5(q) = (1, 0, 1, 1, 2, 2, 3, 3, 5, 5, 7, 7, 9, 9, 10, 9, 8, 6, 4, 2, 1)$$

For $a_6(q)$ we obtain

$$(1, 0, 1, 1, 2, 2, 4, 3, 6, 6, 9, 9, 14, 13, 19, 19, 25)$$

$$25, 33, 30, 37, 34, 36, 31, 31, 21, 18, 10, 6, 2, 1)$$

and for $a_7(q)$ one obtains

$$(1, 0, 1, 1, 2, 2, 4, 4, 6, 7, 10, 11, 16, 17, 23, 26, 33, 37, 46, 51, 62, 69, 81, 89)$$

$$103, 111, 124, 131, 141, 144, 148, 144, 139, 126, 111, 91, 70, 49, 31, 17, 8, 3, 1)$$
Finally, for $a_8(q)$ one obtains:

$$(1, 0, 1, 1, 2, 2, 4, 4, 7, 7, 11, 12, 18, 19, 27, 30, 40, 44, 58, 64, 82, 90, 112, 124$$
$$152, 166, 200, 219, 259, 281, 328, 353, 406, 432, 487, 513, 566, 584, 629, 635$$
$$663, 648, 651, 609, 581, 511, 455, 368, 298, 213, 151, 90, 53, 24, 11, 3, 1)$$

4. The tori-jungle for $S_n$

The field $K_{2,n}$ can be described as a field of lattice invariants under the symmetric group. Consider the permutation $S_n$-lattices

$$V_n = \mathbb{Z}v_1 \oplus \ldots \oplus \mathbb{Z}v_n$$
$$U_n = \oplus_{i,j} \mathbb{Z}u_{ij} \oplus V_n$$

with action induced by $\sigma(v_i) = v_{\sigma(i)}$ and $\sigma(u_{ij}) = u_{\sigma(i)\sigma(j)}$. We then have an $S_n$ morphism from $U_n$ to $V_n$ given by sending $u_{ij}$ to $v_i - v_j$ and $v_i$ to 0 and one verifies that this map fits in an exact sequence

$$0 \rightarrow T_n \rightarrow U_n \rightarrow V_n \rightarrow \mathbb{Z} \rightarrow 0$$

Procesi and Formanek (e.g. [7]) show that $K_{2,n} = \mathbb{F}(T_n)^{S_n}$ the fixed field under the extended $S_n$-action to the group algebra of the kernel $T_n$. This fact links the study of the center of the generic division algebra to the vast theory of lattice- and tori-invariants, see for example [11], [3]. As a conceptual tool in these investigations we introduce the tori-jungle:

**Definition:** The tori-jungle of a finite group $G$ is the picture obtained by classifying all isoclasses of $G$-lattices (i.e. free $\mathbb{Z}$-modules of finite rank with $G$-action) according to their $\mathbb{Z}$-rank (by the Jordan-Zassenhaus theorem there are only finitely many such classes for any given rank) and drawing an edge between $[M]$ and $[N]$ iff there is a $G$-exact sequence

$$0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$$

where $P$ is a permutation lattice.
Of course, by transitivty of the relations, we may restrict attention to edges corresponding to sequences with $P$ a transitive permutation lattice. The $G$-jungle has the following remarkable property: let $l$ be an $\mathbb{F}$-field with a faithful $G$-action by $\mathbb{F}$-automorphisms then the tori-invariants of a $G$-lattice $M$ is the fixed field $l(M)^G$ under the composite $G$-action. One can then study the equivalence classes of tori-invariants under stable equivalence over $l^G$. It turns out that two $G$-lattices $M$ and $N$ have $l^G$-stable equivalent $G$-tori-invariants iff the isoclasses $[M]$ and $[N]$ belong to the same bush in the $G$-jungle. A similar (but weaker) result holds for lattice-invariants (i.e. the fields $\mathbb{F}(M)^G$): if $[M]$ and $[N]$ belong to the same bush in the $G$-jungle then their lattice invariants are stable equivalent over $\mathbb{F}$. Hence, in checking stable rationality of a particular lattice invariant (e.g. the center of the generic division algebra) it is allowed to change the lattice within the same bush (e.g. take one of the roots of minimal rank) and check stable rationality of this simpler situation. This is the Formanek strategy in his proof of rationality in the 4 by 4 case: he reduces the problem of the lattice $T_4$ (which has rank 17) to that of a rank 3 lattice for which he gives a transcendence basis for the lattice-invariants.

If one wants to apply this strategy systematically one needs an invariant to determine whether two lattices lie in the same bush. Luckily, such an invariant exist. A $G$-lattice $F$ is said to be flasque if every short exact sequence $0 \rightarrow P \rightarrow A \rightarrow F \rightarrow 0$ with $P$ a permutation lattice splits. Colliot-Thélène and Sansuc [3] proved that any lattice has a flasque resolution $0 \rightarrow M \rightarrow P \rightarrow F = \phi(M) \rightarrow 0$ with $P$ permutation and $F$ flasque. One can then define the Colliot-semigroup $Col(G)$ which are the stable permutation classes (i.e. after adding permutation lattices to both lattices they become isomorphic) of flasque lattices. A major result of [3] is that $[M]$ and $[N]$ lie in the same bush (and hence have stably equivalent tori- or lattice invariants) iff $\phi(M)$ and $\phi(N)$ determine the same element in $Col(G)$. This semigroup is in general not cancellative but one can write it as a disjoint union of cancellative sub-semigroups which are genus closed [15] (dually, lattices have coflasque resolutions leading to the coflasque semigroup $Cof(G)$ and an invariant $\kappa(M)$ independent on the chosen flasque resolution). Of particular importance is the subgroup of invertible elements of $Col(G)$ which is called the permutation classgroup of $G$ and for which we have a pretty good picture, see e.g. [6]. Snider’s remark [8] really comes down to the fact that if $n$ is not square-free then $\phi(T_n)$ cannot lie in the permutation classgroup of $S_n$. On the other hand, if $n = p$ is a prime number one can show very easily the following result which implies retract rationality (see also [4] for a similar but more geometrically based proof):
Theorem 6: If \( n = p \) a prime number, then the center of the generic division algebra \( \Delta_{2,n} \) is stable equivalent to the lattice invariants of an invertible \( S_n \)-lattice.

Proof: We can split up the long exact sequence determining \( T_n \) in two shorter ones \( 0 \to T_n \to U_n \to I_n \to 0 \) and \( 0 \to I_n \to V_n \to \mathbb{Z} \to 0 \). From the first and [3] we recover that \( \phi(T_n) = \phi(\kappa(I_n)) \) so we are done if we can prove that \( \kappa(I_n) \) is an invertible \( S_n \)-lattice. Invertibility can be checked ring-locally (i.e. localizing at all primes dividing the order of the group) or group-locally (i.e. by restriction to Sylow subgroups). For any prime \( q < p \) let \( G_q \) be a \( q \)-Sylow subgroup of \( S_n \), then the second sequence splits as \( \mathbb{Z}_q G_q \)-lattices, i.e. \( I_n \) (and hence \( \kappa(I_n) \)) is an invertible \( G_q \)-lattice. So, we only need to check invertibility when restricted to a \( p \)-Sylow subgroup of \( S_n \) which is \( \mathbb{Z}/p\mathbb{Z} \). But then \( \kappa(I_n) \) is a coflasque \( \mathbb{Z}/p\mathbb{Z} \)-lattice and hence invertible, done.

In fact one can show more: \( \kappa(I_n) \) determines a torsion element in the permutation class group of \( S_n \) and it must be tractable to check for small values of \( p \) (say 5 or 7) that it is actually the zero-element. As we are not (yet) able to prove this we ask the following

Question': Does the \( S_n \)-lattice \( T_n \) lie in the bush of the permutation lattices whenever \( n = p \) a prime number?

An immediate consequence would be that the corresponding fields \( K_{2,p} \) would be stably rational. In view of the foregoing proof it is clear that we can restrict attention to the special lattice \( I_n \) (of rank \( n - 1 \)) which is just the classical lattice \( A_{n-1} \) see e.g. [5].

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