Stable Rationality of certain Moduli Spaces

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Abstract
In this paper we study the (stable) rationality problem of quotient varieties of a vector space under an almost free action of a reductive quotient of $SL_n$ by reducing it to integral and modular representation theory of the symmetric group $S_n$. In particular, we prove stable rationality of $PGL_5$ and $PGL_7$ quotients.

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1 Introduction

Let $\mathcal{C}$ be an algebraically closed field of characteristic zero and $G$ a reductive linear algebraic group. Let $X$ be a finite dimensional vectorspace on which $G$ acts almost freely, i.e. the stabilizer of a generic point is trivial. We want to investigate stable rationality of the quotient variety $X/G$ i.e. whether the fixed field $\mathcal{C}(X)^G$ is stable rational over $\mathcal{C}$. Bogomolov [6] has shown that this question is independent of the particular choice of $X$.

Moreover, Saltman [27] has indicated how this problem can be reduced to checking stable rationality of certain fields of twisted multiplicative invariants under the corresponding Weyl group. So, let $T$ be a maximal torus in $G$ with normalizer $N_G(T)$ then the Weyl group of $G$ is the finite group $W = N_G(T)/T$. Then, consider the character group $X(T) = Hom_{alg}(T, G^*)$ as a lattice over the integral groupring $\mathbb{Z}W$ and take a permutation $\mathbb{Z}W$-lattice $P$ (i.e. a $\mathbb{Z}W$-lattice having a basis which is permuted under the action of $W$) fitting into a $\mathbb{Z}W$-exact sequence

$$0 \to M \to P \to X(T) \to 0$$

(1)

Saltman [27,Cor.2.7] shows that $\mathcal{C}(X)^G$ is stable equivalent to the field of twisted multiplicative invariants $\mathcal{C}_\alpha(M)^W$ where $\alpha$ is some extension of $M$ by $\mathcal{C}^*$ and in many cases (as the ones we are interested in) the twisting by $\alpha$ can be dispensed with.

Let us now concentrate on the special case where $G$ is a quotient of $SL_n$. Note that the center of $SL_n$ is cyclic of order $n$ and for each divisor $r$ of $n$ let $C_r$ be the unique central subgroup of $SL_n$ which is cyclic of order $r$. If we define $SL_n(r) = SL_n/C_r$ then clearly $SL_n(n) = PGL_n$ and we have epimorphisms

$$SL_n \quad \rightarrow \quad PGL_n$$

$$SL_n(r) \quad \rightarrow$$

(2)

Then, the symmetric group on $n$ letters, $S_n$, is the Weyl group of all these groups and the character lattices of the corresponding maximal tori can be described as follows : let $A_{n-1}$ be the classical root lattice consisting of all integer vectors $(x_1, ..., x_n) \in \mathbb{Z}^{bn}$ such that $\sum_{i=1}^n x_i = 0$. Then $A_{n-1}$ is the character lattice of a maximal torus in $PGL_n$ whereas its dual $A^*_{n-1}$ is that of a maximal torus in $SL_n$ and the Coxeter lattice

$$A_{n-1}[s] = \bigcup_{i=0,r,2r,...,n-r}(i + A_{n-1})$$

(3)
where \( r,s = n \) and for \( i + j = n \)

\[
[i] = \frac{1}{n} (i, \ldots, i, -j, \ldots, -j)
\]

is the character lattice of a maximal torus of \( SL_n(r) \). Saltman realized these \( \mathbb{Z}S_n \)-lattices as epimorphic images of the following permutation lattices: Let \( S_{n-1} \) be embedded in \( S_n \) as the stabilizer of \( u_1 \) in the standard permutation representation \( U_n = \mathbb{Z}u_1 \oplus \ldots \oplus \mathbb{Z}u_n \) of \( S_n \). Let \( V_n \) be the \( \mathbb{Z} \)-lattice on the non-diagonal entries of an \( n \) by \( n \) matrix \( V_n = \mathbb{Z}y_{12} \oplus \mathbb{Z}y_{13} \oplus \ldots \oplus \mathbb{Z}y_{n-1,n} \) and turn it into a permutation \( \mathbb{Z}S_n \)-lattice via the action \( \sigma(y_{ij}) = y_{\sigma(i)\sigma(j)} \).

We now construct the following diagram

\[
\begin{array}{c}
0 \to G_{n,n} \to V_n \xrightarrow{\xi} A_{n-1} \to 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \to G_{n,r} \to V_n \oplus \mathbb{Z}S_n/S_{n-1} \xrightarrow{\xi} A_{n-1}[r] \to 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \to G_{n,1} \to V_n \oplus \mathbb{Z}S_n/S_{n-1} \oplus \mathbb{Z} \xrightarrow{\psi} A_{n-1}^* \to 0
\end{array}
\]

where \( \psi \) extends \( \xi \) extends \( \phi \) and are defined by: \( \phi(y_{ij}) = u_i - u_j \) (note that \( A_{n-1} \) is the kernel of the augmentation map on \( U_n \)); \( \xi \) sends the canonical generator of \( \mathbb{Z}S_n/S_{n-1} \) to \( \frac{1}{r} ((u_2 - u_1) + (u_3 - u_1) + \ldots + (u_n - u_1)) \) and \( \psi \) sends the canonical generator of the second \( \mathbb{Z}S_n/S_{n-1} \) factor to \( \frac{1}{n} ((u_2 - u_1) + (u_3 - u_1) + \ldots + (u_n - u_1)) \). Now, \( \psi \) restricted to this second factor is an epimorphism with kernel the trivial lattice showing that \( G_{n,1} \) is the permutation \( \mathbb{Z}S_n \)-lattice \( V_n \oplus \mathbb{Z}S_n/S_{n-1} \oplus \mathbb{Z} \). Using this terminology we have

**Theorem 1 (Saltman, Prop. 3.1)** Let \( X \) be a vectorspace on which \( SL_n(r) \) acts almost freely. Then, \( G(X)_r \) \( SL_n(r) \) is stable equivalent to the field of lattice invariants \( G(G_{n,r})^{S_n} \). Moreover, both fields are stable equivalent to the center of the generic division algebra of degree \( n \) and exponent \( r \).

In particular, if \( PG_2 \) acts almost freely on a vectorspace \( X \), then \( G(X)_r \) \( PG_2 \) is stable equivalent to \( G(G_{n,n})^{S_n} \) which has the center of the generic division algebra of degree \( n \) as a rational field extension. Rationality of this field of lattice invariants for \( n = 2 \) was known already in the last century and Formanek proved rationality if \( n = 3 \) or \( 4 \) cf. [14],[15]. The stable
rationality problem for $\text{PGL}_n$ quotients arises naturally in many seemingly unrelated problems. Let us give a few examples:

- **Merkurjev-Suslin result**: Using a result of Bloch [5], Procesi [24] showed that stable rationality of $\mathcal{C}(X)^{\text{PGL}_n}$ for all $n$ would imply that the Brauer group of any $\mathcal{C}$-field is generated by cyclic algebras.

- **Vector bundles**: The functionfield of the moduli space of stable rank $n$ vectorbundles over the projective plane $\mathbb{P}_2$ with Chern-numbers $c_1 = 0$ and $c_2 = n$ is a rational field over $\mathcal{C}(G_{n,n})^{S_n}$, cfr. [19].

- **$m$-Subspace problem**: Consider the action of $\text{GL}_1$ on $\text{Grass}(k_1,l) \times \ldots \times \text{Grass}(k_m,l)$. If this action has a stable point then the functionfield of the quotient variety is stable equivalent to $\mathcal{C}(X)^{\text{PGL}_n}$ where $n$ is the g.c.d. of the $k_i$ and $l$, cfr. [20].

In case $n$ is even, $\mathcal{C}(G_{n,2})^{S_n}$ is also of some importance. Let $\text{PO}_n$ be the projective orthogonal group and $\text{PSP}_n$ the projective symplectic group. Then, $\mathcal{C}(G_{n,2})^{S_n}$ is stable equivalent to the functionfields of the quotient varieties of an almost free vectorspace action of $\text{PO}_n$ or $\text{PSP}_n$ see [27, Cor. 3.4].

In this paper we aim to study stable rationality of the fields of lattice invariants $\mathcal{C}(G_{n,r})^{S_n}$ with special emphasis to the case $\mathcal{C}(G_{p,p})^{S_p}$ for $p$ a prime number. The starting point is the theory of tori-invariants as developed in the early seventies by a.o. Endo and Miyata [13], Lenstra [21], Voskresenskii [29] and Colliot-Thélène and Sansuc [7].

Recall that a $\mathbb{Z}W$-lattice ($W$ an arbitrary finite group) $M$ is said to be flasque (resp. coflasque) if $\check{H}^{-1}(H,M) = 0$ (resp. $\check{H}^1(H,M) = 0$) for all subgroups $H$ of $W$. Every lattice $M$ has a coflasque resolution

$$0 \to Q \to P \to M \to 0 \quad (6)$$

with $Q$ coflasque and $P$ permutation. An explicit resolution is obtained by taking

$$P = \bigoplus_{H<W} \mathbb{Z}W/H \otimes_{\mathbb{Z}} M^H \quad (7)$$

with $M^H$ the $H$-invariant elements given trivial $G$-action and the sum taken over all conjugacy classes of subgroups $H$ of $G$. Moreover, if $Q_1$ and $Q_2$ are end terms of coflasque resolutions of $M$ then there exist permutation $\mathbb{Z}W$-lattices $P_i$ s.t. $Q_1 \oplus P_1 \cong Q_2 \oplus P_1$. Hence, introducing the Abelian semigroup
$Cofl(W)$ of stable permutation classes of coflasque $ZZW$-lattices gives a well defined map

$$\kappa : ZZW - \text{lattices} \rightarrow Cofl(W)$$  \hspace{1cm} (8)

assigning to a lattice the class of an end term of a coflasque resolution.

Dually, every $ZZW$-lattice $M$ has a flasque resolution

$$0 \rightarrow M \rightarrow P \rightarrow F \rightarrow 0$$  \hspace{1cm} (9)

with $P$ permutation and $F$ flasque and introducing $Flas(W)$ to be the Abelian semigroup of stable permutation classes of flasque $ZZW$-lattices gives a well defined map

$$\phi : ZZW - \text{lattices} \rightarrow Flas(W)$$  \hspace{1cm} (10)

assigning to a lattice the class of an end term of a flasque resolution.

Now, if $L$ is a $G$-field with a faithful $W$-action we can define for every $ZZW$-lattice $M$ the field of tori-invariants $L(M)^W$. Crucial for our purposes is the following characterization of stable equivalence classes of tori-invariants and its consequence for lattice-invariants:

**Theorem 2 (Colliot-Thélène,Sansuc)** For $ZZW$-lattices $M$ and $N$ we have

- $L(M)^W$ is stable equivalent to $L(N)^W$ over $L^W$ if and only if $\phi(M) = \phi(N)$ in $Flas(W)$

- If $M$ and $N$ are faithful with $\phi(M) = \phi(N)$ in $Flas(W)$ then $G(M)^W$ is stable equivalent to $G(N)^W$ over $G$

The main aim of this paper is therefore to study the place of $\phi(G_{n,r})$ in $Flas(S_n)$.

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2 Some known results.

In this section we summarize the present knowledge on $C(G,n,n)^{S_n}$ apart from rationality for $n \leq 4$ mentioned before.

The moduli space $M(n; 0, n)$ of stable rank $n$-vectorbundles over $IP_2$ with Chern-numbers $c_1 = 0$ and $c_2 = n$ has as its function field a rational field extension of $C(U_n \oplus G_{n,n})^{S_n}$ see e.g. [22,19]. So, the natural approach is to prove that the field of tori-invariants $C(U_n)(G_{n,n})^{S_n}$ is stable rational over $C(U_n)^{S_n}$ which is the rational field on the symmetric functions in the $u_i$.

Barth [2] proved that stable vectorbundles over $IP_2$ with $c_1 = 0$ are classified by their curve of jumping lines in the dual plane $IP_2^*$ (i.e. those lines $l$ s.t. $E \mid l \neq \Omega^{S_n}$) and a theta divisor on this curve. If we take a point $x$ in $IP_2$ then all lines through $x$ form a $IP_1$ in $IP_2$. So, for a sufficiently general vectorbundle $E$ this line will intersect the curve of jumping lines in $n$ distinct (unordered) points defining a natural map

$$M(n; 0, n) \rightarrow IP_1 \times \ldots \times IP_1 / S_n$$

(11)

Maruyama [22] claims that this map induces a stably rational field extension. Unfortunately, his proof breaks down because of the false alleged $PGL(n)$-invariance of the map in [22,p.87].

Using the Colliot-Thélène and Sansuc result mentioned above this approach is equivalent to deriving an exact $ZZS_n$-sequence

$$0 \rightarrow G_{n,n} \rightarrow P_1 \rightarrow P_2 \rightarrow 0$$

(12)

with $P_1$ and $P_2$ permutation $ZZS_n$-lattices. For, stable rationality of the field of tori-invariants $C(U_n)(G_{n,n})^{S_n}$ is equivalent to $\phi(G_{n,n}) = 0$ i.e. $\phi(G_{n,n})$ being stable permutation. By 1980 it was already common knowledge among ringtheorists studying the center of the generic division algebras, e.g. [15] and [24] that this could not be true in general. For the readers convenience we will recall here an amplification of this result and its cohomological proof due to Saltman [26]:

**Theorem 3 (Snider, Saltman)** If $n$ is not squarefree, then there does not exist an exact $ZZS_n$-sequence

$$0 \rightarrow G_{n,n} \rightarrow P \rightarrow Q \rightarrow 0$$

(13)

with $P$ permutation and $Q$ coflasque.
Proof: If \( n = p^2 \cdot m \), then \( S_n \) contains a subgroup \( G \) which is the direct product of a cyclic group of order \( p \) and one of order \( p \cdot m \) such that the action on \( n \) letters is the product action. Restricting any permutation \( \mathbb{Z}S_n \)-lattice \( P \) down to \( G \) we can write it as \( \oplus_i \mathbb{Z}G/H_i \) for some subgroups \( H_i \) of \( G \). But then by Shapiro's lemma: \( \mathcal{H}^2(G, P) = \oplus_i \text{Hom}(H_i, \mathbb{Q}/\mathbb{Z}) \). Thus from the existence of the required sequence we would have

\[
0 \to \mathcal{H}^2(G, G_{n,n}) \to \oplus_i \text{Hom}(H_i, \mathbb{Q}/\mathbb{Z})
\]  \hspace{1cm} (14)

whence any element of \( \mathcal{H}^2(G, G_{n,n}) \) must have order dividing \( p \cdot m < n \). However, using that \( V_n \) and \( U_n \) are free \( \mathbb{Z}G \)-lattices we have that \( \mathcal{H}^2(G, G_{n,n}) = \mathcal{H}^1(G, A_{n-1}) = \mathbb{Z}/n\mathbb{Z} \), a contradiction. \( \square \)

However, things change drastically if \( n = p \) is a prime number. Motivated by the retract rationality result of Saltman [25], Colliot-Thélène and Sansuc proved that such a sequence does exist, even with \( Q \) an invertible \( \mathbb{Z}S_p \)-lattice. We will give a short proof of this result based on the following characterization of invertible lattices which can be found in [7] and [3]

Lemma 1 For a \( \mathbb{Z}W \)-lattice \( M \) the following are equivalent

- \( M \) is an invertible \( \mathbb{Z}W \)-lattice
- \( M \otimes \mathbb{Z}_p \) is an invertible \( \mathbb{Z}_p W \)-lattice for all primes \( p \) dividing the order of \( W \)
- \( M \downarrow_S \) is an invertible \( \mathbb{Z}S \)-lattice for all Sylow subgroups \( S \) of \( W \)
- \( M \downarrow_S \otimes \mathbb{Z}_p \) is a permutation \( \mathbb{Z}_p S \)-lattice for all Sylow \( p \)-subgroups \( S \) of \( W \) (for all \( p \) dividing the order of \( W \))

The following result will be the starting point for our further investigation:

Lemma 2 For all prime numbers \( p \) we have

1. \( \kappa(A_{p-1}) \) is an invertible \( \mathbb{Z}S_p \)-lattice
2. In a coflasque resolution of \( A_{p-1} \) the permutation lattice can be taken to contain only factors \( \mathbb{Z}S_p/H \) where \( p \) does not divide the order of \( H \)
**Proof:** Consider a coflasque resolution of \( A_{p-1} \) as an \( S_p \)-lattice

\[
0 \to \kappa(A_{p-1}) \to P \to A_{p-1} \to 0
\]

For all primes \( q < p \) the epimorphism \( \mathbb{Z}_q \otimes U_p \to \mathbb{Z}_q \) splits entailing that \( \mathbb{Z}_q \otimes A_{p-1} \) is an invertible \( \mathbb{Z}_q S_p \)-lattice. Then, by tensoring the flasque resolution above with \( \mathbb{Z}_q \) also \( \mathbb{Z}_q \otimes \kappa(A_{p-1}) \) is invertible. Applying the foregoing result we have that \( \kappa(A_{p-1}) \) is invertible when restricted to a \( q \)-Sylow subgroup of \( S_p \). Further, the \( p \)-Sylow subgroup of \( S_p \) is cyclic and thus the restriction of the coflasque lattice \( \kappa(A_{p-1}) \) is invertible. The foregoing result finishes the proof.

(2): Consider the explicit description (7) of the middle term of a coflasque resolution. Now let \( H \) be a subgroup of \( S_p \) such that \( p \) divides its order. Then, \( H \) contains a \( p \)-cycle and we have that \( U_H^P = \mathbb{Z} \langle \sum_{i=1}^p u_i \rangle \) from which we deduce that \( A_{p-1}^H = 0 \), done. \( \square \)

As an immediate consequence of this result we get a short proof of the Colliot-Thélène and Sansuc result [8]:

**Theorem 4 (Colliot-Thélène,Sansuc)** For \( p \) a prime number, there does exist an exact sequence of \( \mathbb{Z} S_p \)-lattices

\[
0 \to G_{p,p} \to P \to I \to 0
\]

with \( P \) a permutation \( \mathbb{Z} S_p \)-lattice and \( I \) an invertible \( \mathbb{Z} S_n \)-lattice (and hence in particular coflasque)

**Proof:** Consider the pullback diagram

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
G_{p,p} & = & G_{p,p} \\
\downarrow & & \downarrow \\
0 & \rightarrow & \kappa(A_{p-1}) \\
\| & & \downarrow \\
0 & \rightarrow & V_p \times_{A_{p-1}} P \\
\downarrow & & \downarrow \\
0 & \rightarrow & V_p \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\end{array}
\]

\[
0 \rightarrow \kappa(A_{p-1}) \rightarrow V_p \times_{A_{p-1}} P \rightarrow V_p \rightarrow 0
\]

\[
0 \rightarrow \kappa(A_{p-1}) \rightarrow P \rightarrow A_{p-1} \rightarrow 0
\]

\[
0 \rightarrow 0
\]
Because $V_p$ is permutation and $\kappa(A_{p-1})$ coflasque we get

$$P \times_{A_{p-1}} V_p \cong \kappa(A_{p-1}) \oplus V_p$$

giving rise to the exact $\mathbb{Z}S_p$-sequence

$$0 \to G_{p,p} \to \kappa(A_{p-1}) \oplus V_p \to P \to 0$$

and since $\kappa(A_{p-1})$ is invertible there exists an invertible lattice $I$ and a permutation lattice $P'$ such that $\kappa(A_{p-1}) \oplus I \cong P'$ leading to an exact sequence

$$0 \to G_{p,p} \to V_p \oplus P' \to I \oplus P \to 0$$

of the required type.

\[\square\]

## 3 Some general results

Given a finite group $W$ and two $\mathbb{Z}W$-lattices $M$ and $N$, we want to develop a method to decide whether or not $\phi(M) = \phi(N)$ in $\text{Flas}(W)$. There are basically three major obstructions:

- **local obstruction**: Are the local invariants $\phi_p(M)$ and $\phi_p(N)$ equal for all primes $p \mid \#W$?

- **Burnside obstruction**: If all the local invariants coincide can we find representatives in $\phi(M)$ and $\phi(N)$ which lie in the same genus?

- **genus obstruction**: Given representatives in $\phi(M)$ and $\phi(N)$ which lie in the same genus, does this imply that $\phi(M) = \phi(N)$?

In the special case of the symmetric groups $S_n$ we can describe these obstructions fairly explicitly:

### 3.1 The local obstruction

For any domain $R$ we define an $RW$-lattice $M$ to be flasque iff $\text{Ext}_W^1(M, RW/H) = 0$ for all subgroups $H$ of $W$. With $\text{Flas}(RW)$ we denote the semigroup of stable permutation equivalence classes of flasque $RW$-lattices.
A first test to distinguish $\phi(M)$ and $\phi(N)$ is to check whether they have the same image under the localization map

$$
\text{loc} : \text{Flas}(W) \to \prod_{p \mid \# W} \text{Flas}(\mathbb{Z}_p W)
$$

(15)

That is, we want to study the local invariants $\phi_p(M) = [\phi(M) \otimes \mathbb{Z}_p] \in \text{Flas}(\mathbb{Z}_p W)$. The local semigroups can be described using ordinary and modular representation theory of $W$ as we have by descent-theory a pullback diagram

$$
\begin{array}{c}
\text{Flas}(\mathbb{Z}_p W) \to \text{Flas}(\overline{\mathbb{Z}}_p W) \\
\downarrow \hspace{1cm} \downarrow \\
\text{Flas}(\mathbb{Q} W) \to \text{Flas}(\overline{\mathbb{Q}}_p W)
\end{array}
$$

(16)

In the $S_n$-case we get a first simplification because the permutation characters generate all :

**Lemma 3** For all $n$ and all primes $p$ we have $\text{Flas}(\mathbb{Z}_p S_n) \cong \text{Flas}(\overline{\mathbb{Z}}_p S_n)$

This reduces the computation of the local invariants to modular representation theory of $S_n$. In the special case of the lattices $G_{n,r}$ a lot of these local invariants can be extracted from the definition diagram (5):

**Lemma 4** For all primes $p$ and $r \mid n$ we have :

- If $p \nmid r$ then $\phi_p(G_{n,r}) = 0$ in $\text{Flas}(\overline{\mathbb{Z}}_p S_n)$
- If $p \mid r \nmid n$ then $\phi_p(G_{n,r}) = \phi_p(G_{n,n})$ in $\text{Flas}(\overline{\mathbb{Z}}_p S_n)$

Before we can describe the local invariants $\phi_p(G_{n,r})$ for $p \mid r$ we have to recall a result of G. James [17] on the indecomposable modules of permutation modules corresponding to Young subgroups.

Let $\alpha = (a_1, \ldots, a_k)$ and $\beta = (b_1, \ldots, b_l)$ be two partitions of $n$ then we define $\alpha \trianglelefteq \beta$ iff for all $i$ we have $\sum_{j=1}^{i} a_j \leq \sum_{j=1}^{i} b_j$. Further, we denote by $P_\alpha$ the permutation representation corresponding to the Young subgroup of $\alpha$ i.e.

$$
P_\alpha = \overline{\mathbb{Z}}_p S_n / S_{a_1} \times \ldots \times S_{a_k}
$$

(17)

Using this terminology we have :

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Proposition 1 (G.D. James) For any partition $\alpha$ of $n$ there exists an indecomposable $\mathbb{Z}_p S_n$-lattice $I_\alpha$ such that for all partitions $\beta$ of $n$ we have

- $P_\beta \cong I_{\alpha_1}^{\beta_1} \oplus \cdots \oplus I_{\alpha_m}^{\beta_m}$ with $\beta \trianglelefteq \alpha_i$ for all $i$ and $I_\beta$ occurs with multiplicity one

- If $I_\alpha = I_\beta$ then $\alpha = \beta$

Using this result we get by induction on the dominance order $\trianglelefteq$ the following

Lemma 5 For all $n$ and all primes $p$ we have:

- All invertible $\mathbb{Z}_p S_n$-lattices of the form $I_\alpha$ are stable permutation lattices

- In particular, all projective $\mathbb{Z}_p S_n$-lattices are stable permutation lattices

With $A^p_{n-1}[r]$ we will denote $A_{n-1}[r] \otimes \mathbb{Z}_p$. Notice that $A^p_{n-1}[r]$ is an indecomposable $\mathbb{Z}_p S_n$-lattice and let $\Omega(A^p_{n-1}[r])$ be its first syzygy:

Lemma 6 For all primes $p$ and all $r \mid n$ we have that $\phi_p(G_{n,r}) = \phi_p(\Omega(A^p_{n-1}[r]))$ in $\text{Flas}(\mathbb{Z}_p S_n)$

Proof: From the defining sequence of $G_{n,r}$ we deduce that $\phi_p(G_{n,r}) = \phi_p(\kappa_p(A^p_{n-1}[r]))$. On the other hand, taking a projective cover gives a sequence

$$0 \to \Omega(A^p_{n-1}[r]) \oplus P' \to P \to A^p_{n-1}[r] \to 0 \quad (18)$$

As the projective modules $P$ is stable permutation we can add a permutation $\mathbb{Z}_p S_n$-lattice to the first two terms yielding a sequence

$$0 \to \Omega(A^p_{n-1}[r]) \oplus P_1 \to P_2 \to A^p_{n-1}[r] \to 0 \quad (19)$$

with $P_i$ permutation. But then, $\phi_p(\Omega(A^p_{n-1}[r])) = \phi_p(\kappa_p(A^p_{n-1}[r]))$, done. □

In the special case that $n = r = p$ we even get:

Lemma 7 $\phi_p(G_{p,p}) = -[\Omega^2(\mathbb{Z}_p)] \in \text{Flas}(\mathbb{Z}_p S_p)$

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Proof: \( \mathbb{Z}_p S_p / S_{p-1} \) is a projective \( \mathbb{Z}_p S_p \)-lattice, so \( A_{p-1}^p = \Omega(\mathbb{Z}_p) \). By the foregoing this gives us

\[
\phi_p(G_{p,p}) = \phi_p(\Omega(A_{p-1}^p)) = \phi(\Omega^2(\mathbb{Z}_p))
\]

But, \( \Omega^2(\mathbb{Z}_p) \) is an invertible \( \mathbb{Z}_p S_p \)-lattice and thus \( \phi(\Omega^2(\mathbb{Z}_p)) = -[\Omega^2(\mathbb{Z}_p)] \in \text{Flas}(\mathbb{Z}_p S_p) \).

3.2 The Burnside obstruction

Suppose we have two \( \mathbb{Z}W \)-lattices such that for all primes \( p | \#W \) we have \( \phi_p(M) = \phi_p(N) \) i.e. we can find \( W \)-sets \( T_p \) and \( T'_p \) (depending on the prime \( p \)) such that

\[
\phi_p(M) \oplus \mathbb{Z}_p T_p \cong \phi_p(N) \oplus \mathbb{Z}_p T'_p
\]

We now ask whether one can globalize this to get \( W \)-sets \( T \) and \( T' \) (independent on the prime \( p \)) such that \( \phi(M) \oplus ZZ T \) lies in the same genus as \( \phi(N) \oplus ZZ T' \) i.e. such that for all primes \( p | \#W \) :

\[
\phi_p(M) \oplus \mathbb{Z}_p T \cong \phi_p(N) \oplus \mathbb{Z}_p T'
\]

The method to visualize this obstruction is a slight variation on an idea of A. Dress [12] and is based on the description of \( b(W) \) the Burnside ring of \( W \). This is the Grothendieck ring constructed from the isomorphism classes of finite \( W \)-sets with addition induced by disjoint union and multiplication induced by the cartesian product with diagonal \( W \)-action, see e.g. [10] or [9].

If \( C(W) \) denotes the set of conjugacy classes of subgroups \( H \) of \( W \), then we have an injective ringmorphism

\[
\beta = (\beta_H)_H : b(W) \rightarrow \prod_{(H) \in C(W)} \mathbb{Z}
\]

(23)

determined by the Burnside-markers \( \beta_H \) assigning to a \( W \)-set \( S \) the number \( \#S^H \) of \( H \)-fixed elements. Dress has shown that the image of \( b(W) \) can be characterized as the set of those elements \( \chi = (\chi(H)) \in \prod \mathbb{Z} \) satisfying the congruence relations

\[
\sum_{(K)} n(H,K) \cdot \chi(H) \equiv 0 \mod \#N_W(H)/H
\]

(24)
where the sum is taken over all conjugacy classes of subgroups $K$ of $N_W(K)$ such that $H \triangleleft K$ with cyclic quotient $K/H$ and

$$n(H, K) = \frac{N_W(K)}{N_W(H) \cap N_W(K)} \cdot \#(K/H)$$

(25)

the last term being the number of generators of the cyclic group $K/H$. For more details on computing this Möbius-like function we refer to [18].

Now, assume $M$ and $N$ are two $ZZW$-lattices satisfying (ref eq:local). Recall that a group $H$ is said to be $p$-hypoelementary if $H/O_p(H)$ is cyclic where $O_p(H)$ is the largest normal $p$-subgroup of $H$. With $Hy_p(W)$ we will denote the set of all $p$-hypoelementary subgroups of $W$ and let $\mathcal{H} = \cup_{p \mid \#W} Hy_p(W)$. We will now define a mapping

$$\chi : \mathcal{H} \to ZZ$$

(26)

assigning to a subgroup $H \in Hy_p(W)$ the value $\#T_p^H - \#S_p^H$ defined by the $p$-local data. Of course, we have to verify that this map is well-defined. If $H \in Hy_p(W) \cap Hy_q(W)$ then $H$ is cyclic with generator say $w$. But then,

$$\#T_p^H - \#S_p^H = \chi_{\phi(M)(w)} - \chi_{\phi(N)(w)} = \#T_q^H - \#S_q^H$$

(27)

where $\chi_V$ denotes the rational character of $V$. Clearly, $\chi$ is invariant on conjugacy classes, thereby defining a partial function

$$\chi : C(\mathcal{H}) \subset C(W) \to ZZ$$

(28)

Using this terminology we have:

**Lemma 8** If $M$ and $N$ have the same local invariants then the Burnside obstruction vanishes iff the partial function $\chi$ defined above can be extended to an element in $\prod_{(H) \in C(W)} ZZ$ belonging to the image of $\beta$.

**Proof** : Assume that $\chi$ can be extended to an element in the image of $\beta$. Then, there exist $W$-sets $S$ and $T$ such that for all $H \in Hy_p(W)$

$$\chi(H) = \#T_p^H - \#S_p^H = \#T^H - \#S^H$$

(29)

which entails by [11] that

$$\mathbb{Z}_p T_p \oplus \mathbb{Z}_p S \cong \mathbb{Z}_p T \oplus \mathbb{Z}_p S_p$$

(30)
But then by adding $\mathbb{Z}_p S \oplus \mathbb{Z}_p T$ to both sides of (21) we obtain

$$\phi_p(M) \oplus \mathbb{Z}_p S \oplus \mathbb{Z}_p T \cong \phi_p(N) \oplus \mathbb{Z}_p T \oplus \mathbb{Z}_p S \oplus \mathbb{Z}_p T$$

(31)

But then, by cancellation and the above isomorphism we get

$$\phi_p(M) \oplus \mathbb{Z}_p S \cong \phi_p(N) \oplus \mathbb{Z}_p T$$

(32)

which is independent of the prime $p$. Thus, $\phi(M) \oplus \mathbb{Z}S$ and $\phi(N) \oplus \mathbb{Z}T$ belong to the same genus.

The other implication is obvious. $\square$

We are not able to get any substantial simplifications in the case of the symmetric groups. Clearly, James' result (Proposition 1) will be very useful in computing the function $\chi$.

### 3.3 The genus obstruction

Assume we can find representatives $\Phi_M \in \phi(M)$ and $\Phi_N \in \phi(N)$ which lie in the same genus. Does this imply that $\phi(M) = \phi(N)$? In general, this cannot be true as large cyclic groups already produce counterexamples. For non-cyclic groups this obstruction is related to cancellative properties of $\text{Flas}(W)$:

**Lemma 9** If $W$ is non-cyclic and $\text{Flas}(W)$ is a cancellative semigroup, then the genus obstruction vanishes.

**Proof:** From [16] we recall that the relation modules for a non-cyclic group form a complete genus-class and that they all have the same $\phi$-invariant. If $\Phi_M$ and $\Phi_N$ lie in the same genus and if $R$ is a relation module, then by Roiter's replacement lemma there is a relation module $R'$ such that

$$\Phi_M \oplus R \cong \Phi_N \oplus R'$$

(33)

Taking the $\phi$-invariants on both sides and using cancellation in $\text{Flas}(W)$ gives $\phi(M) = \phi(N)$, done. $\square$

Unfortunately, $\text{Flas}(W)$ is seldom cancellative so we have to find another approach.
Let $K_0(ZW)$ be the Grothendieck group of finitely generated projective $ZW$-modules. By a result of R. Swan we know that all projective $ZW$-modules are locally free entailing that

$$K_0(ZW) \cong \mathbb{Z} \times \text{Cl}(ZW)$$  \hspace{1cm} (34)

where for any $\mathbb{Z}$-order $\Lambda$ in a finite dimensional semisimple $Q$-algebra $\Delta$ the locally free classgroup $\text{Cl}(\Lambda) \subset K_0(\Lambda)$ is defined to be

$$\text{Cl}(\Lambda) = \{ [P] - [P'] \in K_0(\Lambda) \mid P_p \cong P'_p \text{ for all primes } p \}$$  \hspace{1cm} (35)

Now, let $\Lambda$ be a maximal order in $QW$ containing $ZW$ and define

$$D(ZW) = \text{Ker}[\text{Cl}(ZW) \to \text{Cl}(\Lambda)]$$  \hspace{1cm} (36)

induced by the inclusion map. Then, $D(ZW)$ is known to be independent of the choice of $\Lambda$ and $\text{Cl}(\Lambda) = \text{Cl}(ZW)/D(ZW)$ is isomorphic to a product of ray classgroups of certain cyclotomic numberfields see e.g. [9]. Oliver [23, Th 7] has shown that

$$D(ZW) \subset \text{Cl}(ZW)$$  \hspace{1cm} (37)

where $\text{Cl}(ZW)$ is the subgroup of $\text{Cl}(ZW)$ generated by the projective left ideals of $ZW$ with trivial $\phi$-invariant. Using this terminology we have

**Lemma 10** If $\text{Cl}(ZW) = \text{Cl}(ZW)$ then the genus obstruction disappears and any f.g. projective $ZW$-lattice is stable permutation.

**Proof:** Let $\Phi_M$ and $\Phi_N$ be in the same genus, then by Roiter's replacement lemma there is a projective left ideal $I$ of $ZW$ such that

$$\Phi_M \oplus ZW \cong \Phi_N \oplus I$$  \hspace{1cm} (38)

and taking the $\phi$-invariants on both sides gives $\phi(M) = \phi(N)$. \hfill $\Box$

In particular, the lemma applies if $D(ZW) = \text{Cl}(ZW)$ as is the case for example if $Q$ is a splitting field for $W$:

**Lemma 11** For $S_n$ the genus obstruction vanishes and all f.g. projective $ZZS_n$-modules are stable permutation.
4 Some Negative Results

In this section we will show that the local obstruction can be used to prove that \( \phi(G_{p,p}) \neq 0 \) if \( p \) is a prime larger than 3. We have seen before (lemma 7) that

\[
\phi_p(G_{p,p}) = -[\Omega^2(\mathbb{Z}_p)] \in \text{Flas}(\mathbb{Z}_p S_p)
\]

(39)

Therefore, we have to show that \( \Omega^2(\mathbb{Z}_p) \) cannot be a stable permutation \( \mathbb{Z}_p S_p \)-lattice. The vertex of \( \Omega^2(\mathbb{Z}_p) \) is equal to the cyclic \( p \)-Sylow subgroup \( G_p = < x = (1, \ldots, p) > \) of \( S_p \). The normalizer of \( G_p \) is a \( p \)-hypoelementary subgroup

\[
N_p = N_{S_p}(G_p) = < x, y : x^p = y^{p-1} = 1, y^{-1} = x^a >
\]

(40)

where \( a \) is a generator of the cyclic group \( IF_p^* \).

Since the \( p \)-Sylow subgroup is cyclic of order \( p \), non-projective indecomposable \( \mathbb{Z}_p S_p \)-lattices and \( \mathbb{Z}_p N_p \)-lattices behave very well with respect to Green correspondence (see e.g. [1] or Feit III,5):

**Lemma 12** There is a one-to-one correspondence between isomorphism classes of indecomposable non-projective \( \mathbb{Z}_p S_p \)-lattices \( M \) and indecomposable non-projective \( \mathbb{Z}_p N_p \)-lattices \( N \) such that

\[
M \downarrow_{N_p} = N \oplus P
\]

\[
N \uparrow^{S_p} = M \oplus P'
\]

where \( P' \) (resp. \( P \)) is a projective \( \mathbb{Z}_p S_p \)- (resp. \( \mathbb{Z}_p N_p \)) lattice.

In particular, the Green correspondent of the \( \mathbb{Z}_p S_p \)-lattice \( \Omega^2(\mathbb{Z}_p) \) is the \( \mathbb{Z}_p N_p \)-lattice \( \Omega^2(\mathbb{Z}_p) \). Now, let \( X_1(\zeta^i) \) be the \( \mathbb{Z}_p N_p \)-lattice of rank one given by the action \( x \to 1 \) and \( y \to \zeta^j \) where \( \zeta \) is a primitive \((p-1)\)-th root of unity reducing to \( a \mod p \). Then, it is easy to see (e.g. using [4,p.189]) that we have:

**Lemma 13** The \( \mathbb{Z}_p N_p \)-lattice \( \Omega^2(\mathbb{Z}_p) \) is of rank one determined by the action \( x \to 1 \) and \( y \to \zeta^{p-2} \) where \( \zeta \) is a primitive \( p-1 \)-th root of unity.

This gives our first negative result:
Theorem 5 \( \phi(G_{p,p}) \neq 0 \) in \( Flas(S_p) \) for primes \( p \geq 5 \)

Proof: Suppose there exist \( S_p \)-sets \( S \) and \( T \) such that
\[
\Omega^2(\hat{\mathbb{Z}}_p) \oplus \hat{\mathbb{Z}}_p S \cong \hat{\mathbb{Z}}_p T
\]
as \( \hat{\mathbb{Z}}_p S_p \)-lattices. Restricting down to \( N_p \) gives us by Green correspondence an isomorphism of \( \hat{\mathbb{Z}}_p N_p \)-lattices
\[
X_1(\zeta^{p-2}) \oplus (\oplus_i \hat{\mathbb{Z}}_p N_p/H_i) \oplus IP \cong (\oplus_j \hat{\mathbb{Z}}_p N_p/K_j) \oplus IP'
\]
where \( IP \) and \( IP' \) are the projective parts and \( H_i \) and \( K_j \) are subgroups of \( N_p \) containing \( C_p \). Clearly, the projective parts can be cancelled showing that \( X_1(\zeta^{p-2}) \) must be a stable permutation \( \hat{\mathbb{Z}}_p N_p \)-lattice which by the above lemma is impossible for \( p \geq 5 \) by looking at the character value on \( y \) which is not an integer.

This result extends the Snider-Saltman vindication of Maruyama’s argument to prime values of \( n \).

The next best strategy is to find a lattice \( M \) of minimal rank such that \( \phi(M) = \phi(G_{p,p}) \). A natural candidate for \( M \) would be \( A_{p-1}^* \) the dual of the root lattice for the following reason: for any \( \mathbb{Z}W \)-lattice \( M \) it is easy to see that \( \phi(M^*) = (\kappa(M))^* \) and from the previous discussions we have obtained that
\[
\phi(G_{p,p}) = \phi(\kappa(A_{p-1})) = -[\kappa(A_{p-1})] \in Flas(S_p)
\]
Thus, if we can prove that \( [\kappa A_{p-1}]^* = -[\kappa A_{p-1}] \) then we would indeed have that \( \phi(A_{p-1}^*) = \phi(G_{p,p}) \). However, we have

Theorem 6 \( \phi(G_{p,p}) \neq \phi(A_{p-1}^*) \) in \( Flas(S_p) \) for all primes \( p \geq 11 \)

Proof: We know that \( \kappa_p(A_{p-1}) = [\Omega^2(\hat{\mathbb{Z}}_p)] \) and thus \( \kappa_p(A_{p-1})^* = [\Omega^{-2}(\hat{\mathbb{Z}}_p)] \). Under Green correspondence \( \Omega^{-2}(\hat{\mathbb{Z}}_p) \) corresponds to the rank one \( \hat{\mathbb{Z}}_p N_p \)-lattice \( X_1(\zeta) \) and as in the above proof we have to exclude that \( X_1(\zeta) \oplus X_1(\zeta^{p-2}) \) can be a stable permutation \( \hat{\mathbb{Z}}_p N_p \)-lattice. This is easy as the only permutation \( \hat{\mathbb{Z}}_p N_p \)-lattice containing a rank one lattice corresponding to a primitive \( p - 1 \)-th root of unity must contain all. And, for \( p \geq 11 \) there are more than two primitive \( p - 1 \)-th roots of unity.

What’s the upshot of all this? We do have:
Proposition 2 For \( p = 5 \) or \( 7 \) all the local invariants \( \phi_q \) of \( A_{p-1}^* \) and \( G_{p,p} \) coincide

Proof: In these cases we do have that

\[
X_1(\zeta) \oplus X_1(\zeta^{p-2})
\]

is a stable permutation \( \mathbb{Z}_p N_p \)-lattice. By Green correspondence this implies that

\[
\Omega^2(\mathbb{Z}_p) \oplus \Omega^{-2}(\mathbb{Z}_p) \oplus \text{IP}
\]

is a stable permutation \( \mathbb{Z}_p S_p \)-lattice where IP is a projective \( \mathbb{Z}_p S_p \)-module. Using that all projectives are stable permutation gives us the wanted \( [(\kappa_p A_{p-1})^*] = -[\kappa_p A_{p-1}] \) in \( \text{Flas}(\mathbb{Z}_p S_p) \) whence \( \phi_p(G_{p,p}) = \phi_p(A_{p-1}^*) \). Equality for the \( q \)-invariants for \( p \neq q \) is easy.

Hence, at least for \( p \leq 7 \) there is some hope to prove stable rationality by reducing the problem to the lattice invariants of the rank \( p - 1 \) lattice \( A_{p-1}^* \). However, for \( p > 7 \) things become uniformly bad.

5 Some Positive Results

There is a remarkable difference in the \( \mathbb{Z}S_n \)-structure of \( G_{n,n} \) when \( n \) is prime or composite:

Proposition 3 1. If \( n \) is composite \( G_{n,n} \) cannot be coflasque

2. For \( p \) prime \( G_{p,p} \) is an invertible \( \mathbb{Z}S_p \)-lattice

Proof: (1) Let \( n = m.k \) with \( m, k > 1 \) and consider the subgroup \( G = S_m \times S_{(k-1)m} \) of \( S_n \) acting in the natural way on the \( n \) elements. From the defining sequence of \( G_{n,n} \) we get the exact sequence

\[
V_n^G \to A_{n-1}^G \to H^1(G, G_{n,n}) \to 0
\]

Now, it is easy to see that

\[
\begin{pmatrix}
(k-1, \ldots, k-1, -1, \ldots, -1)
\end{pmatrix}_{m} \in A_{n-1}^G
\]

(47)
and that the image of $V^G_n$ under $\pi^G$ consists of the vectors

$$\mathbb{Z} \left( \frac{m(k-1), \ldots, m(k-1), -m, \ldots, -m}{m} \right)_{m(k-1)m} \quad (48)$$

and therefore $H^1(G, G_{n,n}) \neq 0$, done.

(2) To verify coflasque-ness it is enough to check that $H^1(G, G_{p,p}) = 0$ for each conjugacy class of a subgroup $G$ of a $q$-Sylow subgroup of $S_p$. Now, if $q \neq p$ we can take $G$ to be a subgroup of $S_{p-1}$ and hence $G$ fixes at least one element of $U_p$. But then it is easy to see that the sequence

$$0 \rightarrow G_{p,p} \rightarrow V_p \rightarrow U_p \rightarrow \mathbb{Z} \rightarrow 0$$

(49)

splits everywhere as $\mathbb{Z}G$-lattices. So, $G_{p,p} \downarrow_G$ is an invertible $\mathbb{Z}G$-lattice and hence $H^1(G, G_{p,p}) = 0$. If $q = p$ then $G = C_p$ can be chosen to permute the base of $U_p$ and so $A_{p-1}^G = 0$ and then the cohomology sequence entails that $H^1(C_p, G_{p,p}) = 0$.

So, $G_{p,p}$ is coflasque and thus $[\kappa(A_{p-1})] = [G_{p,p}] \in Flas(S_p)$. Using that $\kappa(A_{p-1})$ is an invertible $\mathbb{Z}S_p$-lattice finishes the proof. □

Therefore, in order to prove that $\phi(G_{p,p}) = \phi(A_{p-1}^*)$ for $p = 5$ or 7 it suffices to verify that

$$G_{p,p} \oplus G_{p,p}^*$$

(50)

is a stable permutation $\mathbb{Z}S_p$-lattice. For, we have the sequence

$$0 \rightarrow A_{p-1}^* \rightarrow V_p \rightarrow G_{p,p}^* \rightarrow 0$$

(51)

and if we have that $G_{p,p} \oplus G_{p,p}^* \oplus \mathbb{Z}S \simeq \mathbb{Z}T$ for some $S_p$-sets $S$ and $T$ we can add $G_{p,p} \oplus \mathbb{Z}S$ to the last two terms and obtain the sequence

$$0 \rightarrow A_{p-1}^* \rightarrow G_{p,p} \oplus V_p \oplus \mathbb{Z}S \rightarrow \mathbb{Z}T \rightarrow 0$$

(52)

yielding that $\phi(G_{p,p}) = \phi(A_{p-1}^*)$.

In the previous section we have seen that all local invariants of $G_{p,p} \oplus G_{p,p}^*$ vanish, so we have to check whether the Burnside obstruction vanishes. Thus, we have to compute the partial function $\chi : C(\mathcal{H}) \rightarrow \mathbb{Z}$ and verify that it lies in the image of the Burnside ring.

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Now, let us first compute $\chi(H)$ for $q$-hypoelementary subgroups $H$ of $S_p$ if $q \neq p$. Working over $\mathbb{Z}_q$ we have by invertibility of $G_{p,p}$ that
\[
G_{p,p}^q \oplus G_{p,p}^{*q} \oplus (U_p^q)^{\oplus 2} \simeq (V_p^q)^{\oplus 2} \oplus \mathbb{Z}_q \oplus 2
\]
and hence for a $q$-hypo $H$ we can compute $\chi(H)$ by
\[
\chi(H) = 2 + 2\# V_p^H - 2\# U_p^H
\]
On the other hand, if $q = p$ we can use the fact that over $\mathbb{Z}_p$ we have
\[
G_{p,p}^p \simeq \Omega^2(\mathbb{Z}_p) \oplus \mathbb{Z}_p S_p/(S_p^{S_2} \times S_2)
\]
and use the explicit description of the stable permutation $\mathbb{Z}_p S_p$-lattice $\Omega^2(\mathbb{Z}_p) \oplus \Omega^{-2}(\mathbb{Z}_p)$ to compute $\chi(H)$ for $p$-hypoelementary subgroups.

Let us concentrate now on the case $p = 5$. We need to have fairly precise information on the conjugacy classes of subgroups of $S_5$ which we summarize in the following table:

<table>
<thead>
<tr>
<th>class</th>
<th>representative</th>
<th>order</th>
<th>length</th>
<th>normalizer</th>
<th>hypo</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>S</td>
<td>2,3,5</td>
</tr>
<tr>
<td>B</td>
<td>$\langle(12)\rangle$</td>
<td>2</td>
<td>10</td>
<td>N</td>
<td>2,3,5</td>
</tr>
<tr>
<td>C</td>
<td>$\langle(12)(34)\rangle$</td>
<td>2</td>
<td>15</td>
<td>L</td>
<td>2,3,5</td>
</tr>
<tr>
<td>D</td>
<td>$\langle(123)\rangle$</td>
<td>3</td>
<td>10</td>
<td>N</td>
<td>2,3,5</td>
</tr>
<tr>
<td>E</td>
<td>$\langle(12),(34)\rangle$</td>
<td>4</td>
<td>15</td>
<td>L</td>
<td>2</td>
</tr>
<tr>
<td>F</td>
<td>$\langle(1234)\rangle$</td>
<td>4</td>
<td>15</td>
<td>L</td>
<td>2,3,5</td>
</tr>
<tr>
<td>G</td>
<td>$\langle(12)(34),(13)(24)\rangle$</td>
<td>4</td>
<td>5</td>
<td>Q</td>
<td>2</td>
</tr>
<tr>
<td>H</td>
<td>$\langle(12345)\rangle$</td>
<td>5</td>
<td>6</td>
<td>P</td>
<td>2,3,5</td>
</tr>
<tr>
<td>I</td>
<td>$\langle(12),(345)\rangle$</td>
<td>6</td>
<td>10</td>
<td>N</td>
<td>2,3,5</td>
</tr>
<tr>
<td>J</td>
<td>$\langle(123),(12)\rangle$</td>
<td>6</td>
<td>10</td>
<td>N</td>
<td>3</td>
</tr>
<tr>
<td>K</td>
<td>$\langle(123),(12)(45)\rangle$</td>
<td>6</td>
<td>10</td>
<td>N</td>
<td>3</td>
</tr>
<tr>
<td>L</td>
<td>$\langle(1234),(12)(34)\rangle$</td>
<td>8</td>
<td>15</td>
<td>L</td>
<td>2</td>
</tr>
<tr>
<td>M</td>
<td>$\langle(12345),(25)(34)\rangle$</td>
<td>10</td>
<td>6</td>
<td>P</td>
<td>5</td>
</tr>
<tr>
<td>N</td>
<td>$S_3 \times C_2$</td>
<td>12</td>
<td>10</td>
<td>N</td>
<td>none</td>
</tr>
<tr>
<td>O</td>
<td>$A_4$</td>
<td>12</td>
<td>5</td>
<td>Q</td>
<td>2</td>
</tr>
<tr>
<td>P</td>
<td>$\langle(12345),(2354)\rangle$</td>
<td>20</td>
<td>6</td>
<td>P</td>
<td>5</td>
</tr>
<tr>
<td>Q</td>
<td>$S_4$</td>
<td>24</td>
<td>5</td>
<td>Q</td>
<td>none</td>
</tr>
<tr>
<td>R</td>
<td>$A_5$</td>
<td>60</td>
<td>1</td>
<td>S</td>
<td>none</td>
</tr>
<tr>
<td>S</td>
<td>$S_5$</td>
<td>120</td>
<td>1</td>
<td>S</td>
<td>none</td>
</tr>
</tbody>
</table>
Using this information one can now describe the Burnside ring of \( S_5 \) as the image of \( \mathbb{Z}^{S_{19}} \) under multiplication on the right by the matrix:

\[
\begin{pmatrix}
120 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
60 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
60 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
40 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & 6 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & 0 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
15 & 3 & 3 & 0 & 1 & 1 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 0 & 4 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 4 & 2 & 1 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
10 & 0 & 2 & 4 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
6 & 0 & 2 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
5 & 3 & 1 & 2 & 1 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
2 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

Next, we compute the values of \( \chi(H) \) for \( H \) a 2- or 3-hypoelementary subgroup of \( S_5 \). Using the above formula we obtain
Moreover, we have over the 5-adic integers that
\[ \Omega^2(\mathbb{Z}_5) \oplus \Omega^{-2}(\mathbb{Z}_5) \oplus \mathbb{Z}_5 S_5/M \simeq \mathbb{Z}_5 S_5/H \]
(56)
yielding the isomorphism
\[ G_{5,5}^5 \oplus G_{5,5}^{*5} \oplus \mathbb{Z}_5 S_5/M \simeq \mathbb{Z}_5 S_5/H \oplus (\mathbb{Z}_5 S_5/N)^{\mathbb{S}^2} \]
(57)
allowing us to compute \(\chi\) on 5-hypoelementary subgroups:

<table>
<thead>
<tr>
<th>G</th>
<th>(\chi(G))</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>-2</td>
</tr>
<tr>
<td>P</td>
<td>0</td>
</tr>
</tbody>
</table>

We can extend this partial function \(\chi\) by unknowns \(\chi(N) = a_1\), \(\chi(Q) = a_2\), \(\chi(R) = a_3\) and \(\chi(S) = a_4\). Then, multiplying this \(\chi\)-vector by the inverse of the Burnside matrix we get an integer valued vector (and hence the Burnside obstruction vanishes) provided we have that \(a_1\) and \(a_2\) are even and \(a_3 \equiv a_4 \mod 2\). Hence, we can take all \(a_i\) to be zero and then we obtain from the above computations:

**Lemma 14** \(G_{5,5} \oplus G_{5,5}^* \oplus \mathbb{Z}_5 S_5/D \oplus \mathbb{Z}_5 S_5/M\) lies in the same genus as \(\mathbb{Z}_5 S_5/H \oplus \mathbb{Z}_5 S_5/I \oplus \mathbb{Z}_5 S_5/J \oplus \mathbb{Z}_5 S_5/K\). Moreover, \(\phi(G_{5,5}) = \phi(A_5^*)\).

22
Proof: The first statement follows from the above computations. Hence, because the genus obstruction always vanishes we have that $G_{S,5} \oplus G_{S,5}'$ lies in the permutation tree. But then, since it is an invertible lattice it has to be stable permutation from which the last statement follows.

Next, we have to find an ad hoc argument to prove (stable) rationality of the lattice invariants $\mathcal{G}(A^*_4)^{S_5}$. Clearly, we have the exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow U_5 \rightarrow A^*_4 \rightarrow 0$$

(58)

entailing that $\mathcal{G}(A^*_4)$ is the quotient field of the algebra $\mathcal{G}[U_5]/(u_1 u_2 u_3 u_4 u_5 - 1)$ and the $S_5$-action is induced by that on $\mathcal{G}(U_5)$. Clearly, the invariant field $\mathcal{G}(U_5)^{S_5}$ is rational on the elementary symmetric functions $\sigma_i$ on the $u_j$. But then, $\mathcal{G}(A_4)^{S_5}$ is rational on the images of the first four elementary symmetric functions. One can also give an explicit transcendency basis of $\mathcal{G}(A^*_4)^{S_5}$ when $\mathcal{G}(A^*_4)$ is viewed as a quintic extension (obtained by adding $\sqrt[5]{a_1 a_2 a_3 a_4}$) of $\mathcal{G}(A_4) = \mathcal{G}(a_1, a_2, a_3, a_4)$ with the $S_4$-action as in the following diagram

<table>
<thead>
<tr>
<th>$a$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(12).a</td>
<td>$1/a_1$</td>
<td>$a_2/a_1$</td>
<td>$a_3/a_1$</td>
<td>$a_4/a_1$</td>
</tr>
<tr>
<td>(12345).a</td>
<td>$a_2/a_1$</td>
<td>$a_3/a_1$</td>
<td>$a_4/a_1$</td>
<td>$1/a_1$</td>
</tr>
</tbody>
</table>

Using this presentation we obtain that:

$$\mathcal{G}(A^*_4)^{S_5} = \mathcal{G}(\frac{\sum a_i + 1}{b}, \frac{\sum a_i^2 + 1}{b^2}, \frac{\sum a_i^3 + 1}{b^3}, \frac{\sum a_i^4 + 1}{b^4})$$

(59)

Summarizing the above discussions we have proved:

**Theorem 7** Let $X$ be a vector space with an almost free action of $PGL_5$. Then, the quotient variety $X/PGL_5$ is stably rational.

In particular, this gives us:

**Corollary 1** The moduli space of stable rank 5 bundles over $IP_2$ with $c_1 = 0$ is stably rational.

Now, let us turn our attention to the case when $p = 7$ : Again, we need rather precise information on the conjugacy classes of subgroups of $S_7$. In this case there are 96 different classes out of which 55 are hypoelementary
subgroups. In the appendix we have collected all the necessary information on these conjugacy classes as well as the Burnside matrix for $S_7$. In the rest of this section we follow the subgroup notations of the appendix.

As mentioned before it is easy to compute $\chi(H)$ of a $q$-hypoelementary subgroup with $q \neq 7$ using

$$\chi(H) = 2 + 2\#V^H_7 - 2\#U^H_7$$  \hspace{1cm} (60)

As $V_7$ (resp. $U_7$) is the permutation representation corresponding to the subgroup of class $[88]$ (resp $[94]$) the values of $\chi$ are easily deduced from the Burnside matrix. We obtain:

<table>
<thead>
<tr>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi(H)$</td>
<td>72</td>
<td>32</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td>18</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>$H$</td>
<td>[13]</td>
<td>[14]</td>
<td>[15]</td>
<td>[7]</td>
<td>[16]</td>
<td>[17]</td>
<td>[18]</td>
<td>[19]</td>
<td>[20]</td>
<td>[21]</td>
</tr>
<tr>
<td>$\chi(H)$</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$H$</td>
<td>[22]</td>
<td>[23]</td>
<td>[8]</td>
<td>[30]</td>
<td>[31]</td>
<td>[32]</td>
<td>[33]</td>
<td>[34]</td>
<td>[35]</td>
<td>[36]</td>
</tr>
<tr>
<td>$\chi(H)$</td>
<td>2</td>
<td>18</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$H$</td>
<td>[24]</td>
<td>[25]</td>
<td>[26]</td>
<td>[27]</td>
<td>[37]</td>
<td>[43]</td>
<td>[44]</td>
<td>[45]</td>
<td>[47]</td>
<td>[48]</td>
</tr>
<tr>
<td>$\chi(H)$</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>$H$</td>
<td>[49]</td>
<td>[50]</td>
<td>[51]</td>
<td>[52]</td>
<td>[54]</td>
<td>[55]</td>
<td>[59]</td>
<td>[64]</td>
<td>[65]</td>
<td></td>
</tr>
<tr>
<td>$\chi(H)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H$</td>
<td>[69]</td>
<td>[74]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi(H)$</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Now, the more challenging job to determine $\chi(H)$ for the three 7-hypoelementary subgroups [28],[29] and [56]. The starting point is the description of the stable permutation $\mathbb{Z}_7[56]$-lattice $\Omega^2_{[56]}(\mathbb{Z}_7) \oplus \Omega^{-2}_{[56]}(\mathbb{Z}_7) = W$ induced up to the $S_7$-level giving:

$$W \uparrow^{S_7} \oplus \mathbb{Z}_7[28] \oplus \mathbb{Z}_7[29] \simeq \mathbb{Z}_7[8] \oplus \mathbb{Z}_7[56]$$  \hspace{1cm} (61)

where for each subgroup $[i]$ we denote by $\mathbb{Z}_7[i]$ the permutation lattice $\mathbb{Z}_7S_7/[i]$. By Green correspondence we know that

$$W \uparrow^{S_7} = \Omega^2(\mathbb{Z}_7) \oplus \Omega^{-2}(\mathbb{Z}_7) \oplus \text{IP}^{\oplus 2}$$  \hspace{1cm} (62)

where $\text{IP}$ is a projective $\mathbb{Z}_7S_7$ lattice with characters

24
By James’ result we know that $IP$ is stable permutation with all transitive permutation factors corresponding to Young subgroups. Hence, the description of $IP$ as a stable permutation character coincides with the description of the corresponding character as a linear combination of the Young-subgroup permutation characters. So, we have to multiply the character of $IP$ with the inverse of the permutation-character matrix:

$$
\begin{pmatrix}
5040 \\
2520 120 \\
1260 120 12 \\
630 90 18 6 \\
840 120 0 0 24 \\
420 80 12 0 12 2 \\
210 50 14 6 6 2 2 \\
140 40 12 0 8 4 0 2 \\
210 60 6 0 24 0 0 0 6 \\
105 35 9 3 12 2 0 0 3 1 \\
35 15 7 3 5 3 1 2 1 1 1 \\
42 20 6 0 12 2 0 0 6 0 0 2 \\
21 11 5 3 6 2 2 0 3 1 0 1 1 \\
7 5 3 1 4 2 0 1 3 1 0 2 0 1 \\
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
\end{pmatrix}
$$

Here, the columns correspond to the cyclics in the above mentioned ordering and the rows correspond to the Young-subgroups in the ordering : $[1],[2],[9],[32],[23],[41],[58],[73],[60],[81],[91],[88],[92],[94],[96]$. The resulting vector being

$$(0, 0, 1, -2, -1, 2, 1, -1, 0, 1, -1, -1, 0, 1, 0)$$

we obtain the following description of the projective $IP$ as a stable permutation lattice

$$IP \oplus \mathbb{Z}_\tau[32]^{\oplus 2} \oplus \mathbb{Z}_\tau[23] \oplus \mathbb{Z}_\tau[73] \oplus \mathbb{Z}_\tau[91] \oplus \mathbb{Z}_\tau[88]$$

$$\simeq \mathbb{Z}_\tau[9] \oplus \mathbb{Z}_\tau[41]^{\oplus 2} \oplus \mathbb{Z}_\tau[58] \oplus \mathbb{Z}_\tau[81] \oplus \mathbb{Z}_\tau[94]$$

25
Next, we use the fact that
\[ G_{7,7}^g \oplus G_{7,7}^* \simeq \Omega^2(\mathbb{Z}_7) \oplus \Omega^{-2}(\mathbb{Z}_7) \oplus \mathbb{Z}_7[92]^\mathbb{B}^2 \quad (64) \]
which allows us in view of the above mentioned facts to obtain a description of \( V = G_{7,7}^g \oplus G_{7,7}^* \) as a stable permutation lattice:

\[ V \oplus \mathbb{Z}_7[9]^\mathbb{B}^2 \oplus \mathbb{Z}_7[41]^\mathbb{B}^2 \oplus \mathbb{Z}_7[58]^\mathbb{B}^2 \oplus \mathbb{Z}_7[81]^\mathbb{B}^2 \oplus \mathbb{Z}_7[94]^\mathbb{B}^2 \oplus \mathbb{Z}_7[28] \oplus \mathbb{Z}_7[29] \]
\[ \simeq \mathbb{Z}_7[56] \oplus \mathbb{Z}_7[8] \oplus \mathbb{Z}_7[92]^\mathbb{B}^2 \oplus \mathbb{Z}_7[32]^\mathbb{B}^2 \oplus \mathbb{Z}_7[23]^\mathbb{B}^2 \oplus \mathbb{Z}_7[73]^\mathbb{B}^2 \oplus \mathbb{Z}_7[91]^\mathbb{B}^2 \oplus \mathbb{Z}_7[88]^\mathbb{B}^2 \]

This allows us to compute the values for \( \chi(H) \) for the three 7-hypoelementary subgroups of \( S_7 \):

<table>
<thead>
<tr>
<th>( H )</th>
<th>28</th>
<th>29</th>
<th>56</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi(H) )</td>
<td>-2</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

This partial function \( \chi \) can be shown to extend to an element in the Burnside ring of \( S_7 \). More precisely, the following combination of \( S_7 \)-sets extends \( \chi \):

\[ + [32] - [34] + [47] + [48] + [53] + [56] - [58] + [59] + [63] - [75] \]

Therefore, we obtain from the foregoing computations:

**Lemma 15** The lattice
\[ G_{7,7}^g \oplus G_{7,7}^* \oplus \mathbb{Z}[2] \oplus \mathbb{Z}[3] \oplus \mathbb{Z}[6]^\mathbb{B}^2 \oplus \mathbb{Z}[10] \oplus \mathbb{Z}[13] \]
\[ \oplus \mathbb{Z}[18] \oplus \mathbb{Z}[28] \oplus \mathbb{Z}[29] \oplus \mathbb{Z}[34] \oplus \mathbb{Z}[58] \oplus \mathbb{Z}[75] \]
lies in the same genus as the permutation \( \mathbb{Z}S_7 \)-lattice
\[ \mathbb{Z}[1] \oplus \mathbb{Z}[4] \oplus \mathbb{Z}[8] \oplus \mathbb{Z}[20] \oplus \mathbb{Z}[23] \oplus \mathbb{Z}[24] \oplus \mathbb{Z}[30] \]
\[ \oplus \mathbb{Z}[32] \oplus \mathbb{Z}[47] \oplus \mathbb{Z}[48] \oplus \mathbb{Z}[53] \oplus \mathbb{Z}[56] \oplus \mathbb{Z}[59] \oplus \mathbb{Z}[63] \]
Moreover, \( \phi(G_{7,7}) = \phi(A_6) \).

26
It is a bit surprising to note that in order to prove that the rank 36 lattice $G_{7,7}$ has the same $\phi$-invariant as the rank 6 lattice $A_6^*$, we need to show that two lattices of rank 13412 lie in the same genus!

As in the $5 \times 5$ case it is easy to verify that $G(\mathbf{A}_6^*)^{St}$ is rational over $C$ and therefore we obtain:

**Theorem 8** Let $X$ be a vectorspace with an almost free action of $PGL_7$. Then, the quotient variety $X/PGL_7$ is stably rational.

In particular, this gives us:

**Corollary 2** The moduli space of stable rank 7 bundles over $IP_2$ with $c_1 = 0$ is stably rational.

**References**


[16] K. Gruenberg: Relation Modules


Appendix

A Subgroups of $S_7$

In this appendix we give the relevant information about conjugacy classes of subgroups of $S_7$. In the following list we give for each conjugacy class the order of a representative, a list of generators for a representative (note however that these generators are often not minimal nor the most elegant ones) and the primes for which the representative is hypoelementary:

SUBGROUPS OF $S_7$

$\begin{array}{llll}
[1] & \text{ORDER} & 1 & \text{GENERATORS} : \text{IDENTITY} \\
[2] & \text{ORDER} & 2 & \text{GENERATORS} : (6,7) \\
[3] & \text{ORDER} & 2 & \text{GENERATORS} : (1,2)(3,4) \\
[4] & \text{ORDER} & 2 & \text{GENERATORS} : (1,5)(3,6)(4,7) \\
[5] & \text{ORDER} & 3 & \text{GENERATORS} : (1,5,2)(3,4,7) \\
[6] & \text{ORDER} & 3 & \text{GENERATORS} : (1,6,4) \\
\end{array}$

\text{ALL HYPO}
[ 7 ] ORDER 5, GENERATORS: (1,4,2,3,5)  ALL HYPO
[ 8 ] ORDER 7, GENERATORS: (1,4,5,3,2,7,6)  ALL HYPO
[ 9 ] ORDER 4, GENERATORS: (2,4), (6,7)  2 HYPO
[10 ] ORDER 4, GENERATORS: (1,2)(3,4), (6,7)  2 HYPO
[11 ] ORDER 4, GENERATORS: (1,3,2,4)(5,6)  ALL HYPO
[12 ] ORDER 4, GENERATORS: (1,4,2,3)  ALL HYPO
[13 ] ORDER 4, GENERATORS: (1,2)(3,4), (1,4)(2,3)  2 HYPO
[14 ] ORDER 4, GENERATORS: (1,2)(3,4), (3,4)(6,7)  2 HYPO
[15 ] ORDER 4, GENERATORS: (1,2)(3,4), (1,4)(2,3)(5,6)  2 HYPO
[16 ] ORDER 6, GENERATORS: (1,5,3), (6,7)  ALL HYPO
[17 ] ORDER 6, GENERATORS: (1,5,2)(3,4,7), (1,2)(3,4)  3 HYPO
[18 ] ORDER 6, GENERATORS: (1,5,2)(3,4,7), (1,7)(2,4)(3,5)ALL HYPO
[19 ] ORDER 6, GENERATORS: (1,5,2)(3,4,7), (1,4)(2,7)(3,5)  3 HYPO
[20 ] ORDER 6, GENERATORS: (1,6,4), (1,4)(2,3)  3 HYPO
[21 ] ORDER 6, GENERATORS: (1,6,4), (2,3)(5,7)  ALL HYPO
[22 ] ORDER 6, GENERATORS: (1,6,4), (1,4)(2,5)(3,7)  3 HYPO
[23 ] ORDER 6, GENERATORS: (1,6,4), (4,6)  3 HYPO
[24 ] ORDER 9, GENERATORS: (1,5,2), (3,4,7)  3 HYPO
[ 25 ] ORDER 10, GENERATORS: (1,4,2,3,5), (6,7)  ALL HYPO
[ 26 ] ORDER 10, GENERATORS: (1,4,2,3,5), (1,3)(2,4)  5 HYPO
[ 27 ] ORDER 10, GENERATORS: (1,4,2,3,5), (1,5)(3,4)(6,7)  5 HYPO
[ 28 ] ORDER 14, GENERATORS: (1,4,5,3,2,7,6), (1,3)(2,6)(4,5)  7 HYPO
[ 29 ] ORDER 21, GENERATORS: (1,4,5,3,2,7,6), (1,4,7)(2,3,6)  7 HYPO
[ 30 ] ORDER 8, GENERATORS: (2,7,5,3), (2,3)(5,7)  2 HYPO
[ 31 ] ORDER 8, GENERATORS: (5,6), (1,4,2,3)  2 HYPO
[ 32 ] ORDER 8, GENERATORS: (1,2), (3,4), (6,7)  2 HYPO
[ 33 ] ORDER 8, GENERATORS: (1,2)(3,4), (1,4)(2,3), (5,6)  2 HYPO
[ 34 ] ORDER 8, GENERATORS: (2,7,4,6)(3,5), (2,6)(3,5)(4,7)  2 HYPO
[ 35 ] ORDER 8, GENERATORS: (1,3,2,4)(5,6), (1,4)(2,3)  2 HYPO
[ 36 ] ORDER 8, GENERATORS: (1,4,2,3), (3,4)(6,7)  2 HYPO
[ 37 ] ORDER 12, GENERATORS: (1,5,3), (2,4), (6,7)  2 HYPO
[ 38 ] ORDER 12, GENERATORS: (1,5,2)(3,4,7), (1,2)(3,4), (1,4)(2,7)(3,5)  NOT
[ 39 ] ORDER 12, GENERATORS: (1,6,4), (2,5)(3,7), (1,4)  NOT
[ 40 ] ORDER 12, GENERATORS: (1,5,3), (1,3)(2,4), (6,7)  NOT
[ 41 ] ORDER 12, GENERATORS: (1,5,3), (3,5), (6,7)  NOT
[ 42 ] ORDER 12, GENERATORS: (1,6,4), (1,4)(2,3), (2,3)(5,7)  NOT
[ 43 ] ORDER 12, GENERATORS : (1,6)(2,7,3,5), (2,7,3,5)(4,6)  3 HYPO
[ 44 ] ORDER 12, GENERATORS : (2,7,3,5), (1,6,4)  2 ALL HYPO
[ 45 ] ORDER 12, GENERATORS : (1,6,4), (2,3)(5,7), (2,7)(3,5)  2 HYPO
[ 46 ] ORDER 12, GENERATORS : (1,6,4), (2,3)(5,7), (1,4)(2,5)(3,7)  NOT
[ 47 ] ORDER 12, GENERATORS : (1,4,2)(5,6,7), (1,2,3)(5,6,7)  2 HYPO
[ 48 ] ORDER 12, GENERATORS : (1,4,2), (1,4,3)  2 HYPO
[ 49 ] ORDER 12, GENERATORS : (1,4,7)(2,3,6), (1,3,6)(2,4,7)  2 HYPO
[ 50 ] ORDER 18, GENERATORS : (1,5,2)(3,4,7), (1,2,5)(3,4,7), (1,7)(2,4)(3,5)  3 HYPO
[ 51 ] ORDER 18, GENERATORS : (1,5,2), (3,4,7), (1,2)(3,4)  3 HYPO
[ 52 ] ORDER 18, GENERATORS : (1,2,5), (3,4,7), (3,4)  3 HYPO
[ 53 ] ORDER 20, GENERATORS : (1,4,2,3,5), (1,3)(2,4), (1,5)(3,4)(6,7)  NOT
[ 54 ] ORDER 20, GENERATORS : (2,4,3,5)(6,7), (1,2,3,4)(6,7)  5 HYPO
[ 55 ] ORDER 20, GENERATORS : (1,4,3,2), (2,5,3,4)  5 HYPO
[ 56 ] ORDER 42, GENERATORS : (1,4,5,3,2,7,6), (1,4,7)(2,3,6), (1,3)(2,6)(4,5)  7 HYPO
[ 57 ] ORDER 16, GENERATORS : (1,3,2,4)(6,7), (1,4,2,3), (1,4)(2,3)  2 HYPO
[ 58 ] ORDER 24, GENERATORS : (1,5,3), (1,3), (2,4), (6,7)  NOT
[ 59 ] ORDER 24, GENERATORS : (2,7,3,5), (1,6,4), (2,7)(3,5)  2 HYPO

32
[ 60 ] ORDER 24, GENERATORS : (2,7,3,5), (2,3,7,5) NOT

[ 61 ] ORDER 24, GENERATORS : (5,6), (5,7), (1,4,2,3) NOT

[ 62 ] ORDER 24, GENERATORS : (5,7,6), (1,2)(3,4), (1,4)(2,3), (5,6) NOT

[ 63 ] ORDER 24, GENERATORS : (2,7,4,6)(3,5), (1,3)(2,6,4,7), (2,6)(3,5)(4,7) NOT

[ 64 ] ORDER 24, GENERATORS : (1,4,7)(2,3,6), (1,3,6)(2,4,7), (1,2)(3,4)(6,7) 2 HYPO

[ 65 ] ORDER 24, GENERATORS : (1,4,2), (1,4,3), (1,2)(3,4)(5,6) 2 HYPO

[ 66 ] ORDER 24, GENERATORS : (1,6)(2,7,3,5), (2,7,3,5)(4,6), (1,4)(2,3) NOT

[ 67 ] ORDER 24, GENERATORS : (2,7,3,5), (1,6,4), (1,4)(2,3) NOT

[ 68 ] ORDER 24, GENERATORS : (1,3,2,4)(5,6), (1,3,4,2)(6,7) NOT

[ 69 ] ORDER 24, GENERATORS : (1,3,2,4)(5,6), (1,4,3,2)(5,6) 2 HYPO

[ 70 ] ORDER 24, GENERATORS : (1,6,2,7)(3,4), (1,3,2,4)(6,7) NOT

[ 71 ] ORDER 24, GENERATORS : (1,4,2,3), (3,7,4,6) NOT

[ 72 ] ORDER 36, GENERATORS : (1,2,5), (3,4,7), (1,2)(3,4), (1,7)(2,4)(3,5) NOT

[ 73 ] ORDER 36, GENERATORS : (1,2,5), (3,4,7), (1,2), (3,4) NOT

[ 74 ] ORDER 36, GENERATORS : (1,7)(2,3,5,4), (1,7,5,4)(2,3) 3 HYPO

[ 75 ] ORDER 36, GENERATORS : (1,6,4)(2,5,7), (1,6,4)(3,5,7) NOT
[ 76 ] ORDER 40, GENERATORS : (2,4,3,5), (6,7), (1,4,3,2) NOT
[ 77 ] ORDER 60, GENERATORS : (1,3,6,4,5), (1,5,3,4,6) NOT
[ 78 ] ORDER 60, GENERATORS : (1,5,7,4,6), (3,4,6,5,7) NOT
[ 79 ] ORDER 48, GENERATORS : (5,6), (5,7), (1,4,2,3), (1,4)(2,3) NOT
[ 80 ] ORDER 48, GENERATORS : (1,7,5,4)(3,6), (4,7), (1,6,5,3) NOT
[ 81 ] ORDER 48, GENERATORS : (5,6), (1,4,3,2), (1,4,2,3) NOT
[ 82 ] ORDER 72, GENERATORS : (1,7)(2,3,5,4), (1,7,5,4)(2,3), (1,7)(2,4)(3,5) NOT
[ 83 ] ORDER 72, GENERATORS : (2,7,3,5), (2,3,7,5), (1,6,4)(2,5,7) NOT
[ 84 ] ORDER 72, GENERATORS : (1,3,2,4)(5,6), (1,4,3,2)(5,6), (1,3,4,2)(6,7) NOT
[ 85 ] ORDER 72, GENERATORS : (1,4,2)(5,6,7), (1,2,4)(5,6,7), (1,3,2)(5,6,7), (1,2)(3,4)(5,6) NOT
[ 86 ] ORDER 120, GENERATORS : (3,7,4,6), (1,6,5,7), (1,6,3,5) NOT
[ 87 ] ORDER 120, GENERATORS : (1,3,5,4)(2,7), (1,3,4,5)(2,7), (1,6,4,3)(2,7) NOT
[ 88 ] ORDER 120, GENERATORS : (3,6,4,5), (1,6,3,5) NOT
[ 89 ] ORDER 120, GENERATORS : (1,3,6,4,5), (1,5,3,4,6), (2,7)(3,4)(5,6) NOT
[ 90 ] ORDER 168, GENERATORS : (1,4,3,2)(5,6), (1,6,2,7)(3,4) NOT
[91] ORDER 144, GENERATORS : \((1,4), (2,7,5,3), (1,6), (2,7,3,5)\) NOT

[92] ORDER 240, GENERATORS : \((1,3,5,4)(2,7), (1,3,4,5)(2,7), \) NOT
\((1,6,4,3)(2,7), (3,6,4,5)\)

[93] ORDER 360, GENERATORS : \((1,3,4,2)(6,7), (1,6,2,7)(3,4)\) NOT

[94] ORDER 720, GENERATORS : \((1,3,4,2)(6,7), (1,6,2,7)(3,4), \) NOT
\((1,4,2,3)\)

[95] ORDER 2520, GENERATORS : \((1,3,2,4)(5,6), (1,2)(3,4,6,5), \) NOT
\((1,3,7,2)(4,5)\)

[96] ORDER 5040, GENERATORS : \((5,6), (1,2)(3,4,6,5), \) NOT
\((1,3,7,2)(4,5), (1,4,2,3)\)

B  Burnside matrix of \(S_7\)

The following figure is the Burnside matrix for \(S_7\) using the notations of the foregoing list. Note that this time the subgroups are ordered in ascending size: