Counterexamples to the Gel’fand-Kirillov
Conjecture
(d’après J. Alev, A. Ooms and M. Van den Bergh)

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These are notes of a talk in the UIA-algebra seminar on the paper ”A class of counter examples to the Gel’fand-Kirillov conjecture” by Jacques Alev, Alfons Ooms and Michel Van den Bergh [1]. In order to outline the key ideas to ringtheorists, we restrict to the case of the non-special group $\text{PGL}_n$ and the invariant-theoretic setting of generic matrices. Some effort was made to include proofs of basic facts on generic matrices.

1 The strategy

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and $U(\mathfrak{g})$ (resp. $D(\mathfrak{g})$) its enveloping algebra (resp. the division ring of fractions).

**Gel’fand-Kirillov conjecture**: For a $\mathbb{C}$-Lie algebra $\mathfrak{g}$, $D(\mathfrak{g}) \simeq D_k(L)$ a Weyl-skewfield with center $L$, a purely transcendental field over $\mathbb{C}$.

**Definition 1** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and $F$ the center of the division ring of fractions $D(\mathfrak{g})$. A division algebra $\Delta$, finite dimensional over its center $F$ is called $\mathfrak{g}$-bad iff there exists a field extension $F \subset F'$ satisfying the following properties:

1. The extended algebra $\Delta \otimes_F F'$ is not a domain.
2. There is an embedding $F \subset F' \subset D(\mathfrak{g})$.

**Theorem 1** If $\mathfrak{g}$ is a Lie algebra admitting a $\mathfrak{g}$-bad division algebra, then $\mathfrak{g}$ is a counterexample to the Gel’fand-Kirillov conjecture.

1.1 A filtered argument

Let $\mathfrak{g} \subset F$ any field and consider the $k$-th Weyl algebra $A_k(F)$ with center $F$. This is the algebra generated by $x_i, y_j$, $1 \leq i \leq k$ with commutation relations

$$[x_i, x_j] = [y_i, y_j] = 0 \text{ and } [x_i, y_j] = \delta_{ij}$$

If we put $\text{deg}(x_i) = \text{deg}(y_j) = 1$, $A_k(F)$ is a filtered algebra with associated graded ring

$$\text{gr}(A_k(F)) = F[x_1, \ldots, x_k, y_1, \ldots, y_k]$$

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For $d \in A_k(F)_i - A_k(F)_{i-1}$ we denote its image in $gr(A_k(F))$, by $\sigma(d)$. Because $gr(A_k(F))$ is a domain, $\sigma$ is multiplicative.

Let $D_k(F)$ denote its quotient ring of fractions which is a division algebra with center $F$. We can extend the filtration on $A_k(F)$ to a $\mathbb{Z}$-filtration on $D_k(F)$ by defining the degree and symbol of a fraction $deg(fg^{-1}) = deg(f) - deg(g)$ and $\sigma(fg^{-1}) = \frac{\sigma(f)}{\sigma(g)}$. Again, the fact that $gr(A_k(F))$ is a commutative domain makes these definitions well-defined and shows that

$$gr(D_k(F)) = Q_{gr}(F[x_1, \ldots, x_k, y_1, \ldots, y_k])$$

the $\mathbb{Z}$-graded ring obtained by inverting all homogeneous elements of $gr(A_k(F))$. Its part of degree zero is a field $L$, in fact it is a purely transcendental field extension of $F$ in $2k - 1$ variables, for example $\{\frac{x_2}{x_1}, \ldots, \frac{x_k}{x_1}, \frac{y_1}{x_1}, \ldots, \frac{y_k}{x_1}\}$. Further, it is then clear that the part of degree $i$ of this graded localization is then $Lx_1^i$. Hence,

$$gr(D_n(F)) = F(x_2, \ldots, y_k, x_1^{-1})$$

**Lemma 1** The filtration degree zero part of $D_n(F)$, $D_0$ is a discrete valuation ring with maximal ideal $D_{-1}$ and residue field $F(\frac{x_2}{x_1}, \ldots, \frac{y_k}{x_1})$.

**Proof:** (compare with [5, Prop. 3.1]) The filtration-degree allows us to define a function

$$v : D_k(F) \rightarrow \mathbb{Z} \cup \{\infty\}$$

by $v(0) = \infty$ and $v(d) = -deg(d)$ for all $0 \neq d \in D_k(F)$. Using the fact that $gr(D_k(F))$ is a commutative domain one readily verifies that $v(dd') = v(d) + v(d')$ and $v(d + d') \geq \min(v(d), v(d'))$ for all $d, d' \in D_k(F)$. Hence, $v$ is a discrete valuation, with valuation ring $D_0$ and maximal ideal $D_{-1}$ and residue field $D_0/D_{-1} = gr(D_k(F))_0$ which is the required purely transcendental field. $\square$

### 1.2 The proof of the theorem

**Proof:** (compare with [1, Prop. 3.1]) Assume that the statement of the conjecture holds for $g$, then there would be a $k \in \mathbb{N}$ such that

$$D(g) \simeq D_k(F)$$

Assume there is a $g$-bad division algebra $\Delta$ with center $F$ and let $F \subset F'$ be the corresponding field extension. Consider the discrete valuation $v$ on $D_k(F)$ considered above and restrict it to the commutative subfield $F'$. Then either of the following two cases occurs:

1. **the induced valuation is trivial.** Then going to the residue field gives the inclusions

   $$F \subset F' \subset F(\alpha_1, \ldots, \alpha_{2k-1})$$

2. **the induced valuation is non-trivial.** Then, there is a discrete valuation ring $R$ with field of fractions $F'$ and residue field $R/m$ with inclusions

   $$F \subset R/m \subset F(\alpha_1, \ldots, \alpha_{2k-1})$$

In the first case we are done. For, consider the division algebra $\Delta$ with center $F$ and tensor it with the purely transcendental field-extension $F(\alpha_1, \ldots, \alpha_{2k-1})$. We obtain

$$\Delta(\alpha_1, \ldots, \alpha_{2k-1})$$
which is still a division algebra, contradicting the fact that for the intermediate algebra we have that $\Delta \otimes_F F'$ is not a domain.

For the second case we can repeat the above argument provided we can show that $\Delta \otimes_F R/m$ is not a division algebra. Choose $0 \neq f \in \Delta \otimes_F F'$ with $f^2 = 0$. As $R$ is a discrete valuation ring of $F'$ with uniformizing parameter say $\pi$ there is a natural number $m$ such that $\pi^m f \in \Delta \otimes R$. Let $l \in \mathbb{Z}$ minimal with this property then $\pi^l f \neq 0$ in $\Delta \otimes R/m$ but still has square zero, finishing the proof. \hfill $\square$

2 The counter example

2.1 Linear algebra and invariant theory

With $X_n$ we will denote the affine space of $n \times n$ matrix couples

$$X_n = M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$$

The group $GL_n(\mathbb{C})$ acts on this space by simultaneous conjugation

$$g.(A,B) = (gAg^{-1}, gBg^{-1})$$

Clearly, the action of the center $\mathbb{C}.I_n \subset GL_n(\mathbb{C})$ is trivial, so we really have a $PGL_n(\mathbb{C})$-action.

**Lemma 2** The set $U_n$ of couples $(A,B)$ which generate $M_n(\mathbb{C})$ as a $\mathbb{C}$-algebra is a Zariski-open $PGL_n$-invariant set in $X_n$. Moreover, the $PGL_n$-stabilizer of any point in $U_n$ is trivial.

**Proof:** (compare with [8, 6.1 and 6.2]) If $A$ and $B$ do not generate $M_n(\mathbb{C})$, then the dimension of the space spanned by successive powers of $A$ and $B$ is $\leq n^2 - 1$ which can be expressed by the vanishing of $n^2 \times n^2$-minors involving polynomials in the coefficients of $A$ and $B$. Hence this set is closed and it suffices to show that the complement is non-empty.

Let $A$ be a diagonal matrix with distinct eigenvalues and let $C_1, \ldots, C_d \in M_n(\mathbb{C})$ which generate $M_n(\mathbb{C})$ as an algebra. Let $S_1, \ldots, S_k$ the list of subspaces of $\mathbb{C}^n$ which are left invariant by $A$ (this list is finite since the eigenvalues are distinct). The $C_i$ do not have a subspace which is simultaneously invariant (as they generate $M_n(\mathbb{C})$).

For every $j$ we can therefore find an $i$ such that $C_i$ does not send $S_j$ into itself and so there is a non-empty Zariski-open subset of $\mathbb{C}^k$

$$V_j = \{(a_1, \ldots, a_k) \in \mathbb{C}^k \mid (a_1 C_1 + \ldots + a_k C_k) S_j \not\subset S_j\}$$

(observe that sending $S_j$ into itself is a closed condition). Take a point $(c_1, \ldots, c_k) \in \bigcap_{j=1}^k V_j$, then $A$ and $B = c_1 C_1 + \ldots + c_k C_k$ do not have a common invariant subspace and hence they generate $M_n(\mathbb{C})$ as an algebra.

Now, take $g \in GL_n(\mathbb{C})$ such that $g$ fixes $(A,B) \in U_n$, that is, $g$ commutes with both $A$ and $B$ and hence with all of $M_n(\mathbb{C})$, so $g$ is central. Hence, the $PGL_n$-stabilizer of $(A,B)$ is trivial. \hfill $\square$

The coordinate ring $\mathbb{C}[X_n]$ is a polynomial ring in $2n^2$ variables

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} \quad Y = \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nn} \end{bmatrix}$$

The action of $PGL_n$ on $X_n$ induces an action by automorphisms on $\mathbb{C}[X_n]$. For example, if $g \in GL_n(\mathbb{C})$ then $g.x_{ij}$ is the $(i,j)$-entry of the matrix $gXg^{-1}$ in
$M_n(\mathbb{C}[X_n])$. This action of $PGL_n$ extends to the function field $\mathbb{C}(X_n)$. We would like to have a concrete description of the fixed field under this action $\mathbb{C}(X_n)^{PGL_n}$.

We need to recall a standard result in invariant-theory known as Rosenlicht’s theorem, see [4, p. 143] or [9, §IV.2] for a proof. In our case it asserts that there is a Zariski-open $PGL_n$-stable subset $U \subset X_n$ such that $\mathbb{C}(X_n)^{PGL_n}$ is the subfield which separates orbits in $U$. Moreover, the transcendence degree of $\mathbb{C}(X_n)^{PGL_n}$ is then $\dim(X_n) - \max_u \dim_{\mathbb{C}}(U, u)$. For a slightly stronger result see [4, §II.3.4].

Define the ring $\mathbb{G}_n$ of generic matrices as the subring of $M_n(\mathbb{C}[X_n])$ generated by the two matrices $X$ and $Y$.

**Lemma 3** The fixed field $\mathbb{C}(X_n)^{PGL_n}$ is the subfield of $\mathbb{C}(X_n)$ generated by the coefficients of the characteristic polynomial of elements in $\mathbb{G}_n$. Moreover, $\text{trdeg}_\mathbb{C}(\mathbb{C}(X_n)^{PGL_n}) = n^2 + 1$.

**Proof:** (compare with [2, p. 560-61]) In view of the action of $PGL_n$ on $\mathbb{C}[X_n]$ it is clear that these coefficients are invariant functions, that is they are contained in $\mathbb{C}[X_n]^{PGL_n}$ and hence in the fixed field. In order to show that they generate $\mathbb{C}(X_n)^{PGL_n}$ it suffices by Rosenlicht’s result to show that they separate distinct orbits in $U_n$.

So, let $(A, B)$ and $(A', B')$ be in $U_n$ such that for all coefficients of characteristic polynomials $c_s(X, Y)$ of elements $s \in \mathbb{G}_n$ we have $c_s(A, B) = c_s(A', B')$. Then, we claim that these points belong to the same orbit.

Take an element $z(X, Y) \in \mathbb{G}_n$ such that $z = z(A, B)$ (and hence also $z' = z(A', B')$) is an $n \times n$ matrix with distinct eigenvalues. Then we can diagonalize $z$ and $z'$. Hence, replacing $(A, B)$ and $(A', B')$ by points in their orbits we may assume that $z = z'$ a diagonal matrix with distinct eigenvalues (this operation already fixes a flag of subspaces of $\mathbb{C}^n$). Suitable polynomials $z_{11}, ..., z_{nn}$ of $z$ can then be found such that

$$z_{ii}(A, B) = z_{ii}(A', B') = e_{ii}$$

where $e_{ij}$ is the matrix with 1 at place $(i, j)$ and zeroes elsewhere.

Further, there are elements $h_{ij} \in \mathbb{G}_n$ such that $h_{ij}(A, B) = e_{ij}$ (because $A$ and $B$ generate $M_n(\mathbb{C})$) and define $z_{ij} = z_{ii}h_{ij}z_{jj}$ then $z_{ij}(A, B) = e_{ij}$ and $z_{ij}(A', B')$ has at most one non-zero entry namely the $(i, j)$ one. Because $1 = \text{tr}(z_{ij}z_{ji}(A, B)) = \text{tr}(z_{ij}z_{ji}(A', B'))$ we know that this $z_{ij}(A', B') \neq 0$.

Conjugating $(A', B')$ by a diagonal matrix (and so going to another point in the orbit, if necessary) we may assume that $z_{ij}(A', B') = e_{1j}$ for all $1 \leq j \leq n$ (this operation fixes a basis in the flag).

But then it is easy to deduce that for all $i, j$ we have $z_{ij}(A', B') = e_{ij}$. From this we can deduce that $(A, B) = (A', B')$. For example

$$A_{ij} = \text{tr}(z_{ii}Xz_{ji}(A, B)) = \text{tr}(z_{ii}Xz_{ji}(A', B')) = A'_{ij}$$

finishing the proof of the claim and the lemma. \qed

Because $PGL_n$ acts as automorphisms on $\mathbb{C}(X_n)$, its Lie algebra $\mathfrak{sl}_n$ acts by derivations on $\mathbb{C}(X_n)$. Recall that $\text{Der}_\mathbb{C}(\mathbb{C}(X_n))$ is the $\mathbb{C}(X_n)$-vectorspace of all $\mathbb{C}$-derivations of $\mathbb{C}(X_n)$ and has dimension $2n^2$.

**Lemma 4** The natural map

$$\mathbb{C}(X_n) \otimes_\mathbb{C} \mathfrak{sl}_n \to \text{Der}_\mathbb{C}(\mathbb{C}(X_n))$$

is injective.
**Proof:** (compare with [1, (2.3)]) Let $x = (A, B)$ be a point in $U_n$, then the orbit-map $\mu : PGL_n \to X_n$ determined by sending $g$ to $g.x$ is injective. Hence so is the differential of the orbit-map

$$(d\mu)_x : T_x(PGL_n) = \text{Lie}(PGL_n) = \mathfrak{sl}_n \to T_x(X_n)$$

see for example [4, lemma p. 75]. This can also be seen directly as this map sends $h$ to $([h, A], [h, B])$ using the natural identification $T_x(X_n) \simeq X_n$.

Now, assume $\sum_j f_j \otimes h_j$ is in the kernel of the natural map with all $h_j$ $\mathbb{C}$-linearly independent elements of $\mathfrak{sl}_n$ and the $f_j$ rational functions on $X_n$. By definition there is a Zariski-open set in $X_n$ where all $f_i$ are determined. So, we can choose a point $x \in U_n$ such that all $f_j$ are defined in $x$ and at least one $f_j(x) \neq 0$. But then the Lie-element $\sum_j f_j(x)h_j$ maps to zero in $T_x(X_n)$ a contradiction. \hfill $\Box$

### 2.2 Two division algebras with center $\mathbb{C}(X_n)^{PGL_n}$

**Lemma 5** The ring of generic matrices $\mathbb{G}_n$ is a domain.

**Proof:** (compare with [3, Th. 22] and [7, Th. III.1.3]) First we claim that $\mathbb{G}_n$ is a prime ring. This follows if we can show that $\mathbb{G}_n \mathbb{C}(X_n) = M_n(\mathbb{C}(X_n))$ which is prime and a central extension of $\mathbb{G}_n$ (which implies that the intersection of a prime ideal with $\mathbb{G}_n$ is prime). In fact, we show that the $n^2$ elements $X^iY^j (0 \leq i, j \leq n)$ span $M_n(\mathbb{C}(X_n))$ as a $\mathbb{C}(X_n)$-vector space. This follows if we can show that

$$\det(\text{tr}((X^iY^j)(X^iY^m))) \neq 0$$

(use the non-degeneracy of the trace). Now, consider the Ore-extension $\Lambda = \mathbb{C}(u)(v, \sigma)$ where $\sigma(u) = u^n$ with $\zeta_n$ a primitive $n$-th root of unity. $\Lambda$ is a division algebra with of dimension $n^2$ over its center $\mathbb{C}(u^n, v^n)$ and a basis is given by the elements $u^i v^j$ for $0 \leq i, j \leq n$, so

$$\det(\text{tr}((u^i v^j)(u^m v^m))) \neq 0$$

In view of the map $\mathbb{G}_n \to \Lambda$ sending $X$ to $u$ and $Y$ to $v$ the above determinant cannot vanish and hence $\mathbb{G}_n$ is prime.

Now, assume that $\mathbb{G}_n$ is not a domain, then there are $a, b \in \mathbb{G}_n$ such that $ab = 0$. As $\mathbb{G}_n$ is prime there is an $r \in \mathbb{G}_n$ such that $f(x, y) = bra \neq 0$ but $f^2 = bra br = 0$.

Since $f(X, Y) \neq 0$ the induced regular map $f : M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is not the zero-map but then the same holds for extended maps $f \otimes L$ for any field-extension $\mathbb{C} \subset L$. However, since $f(X, Y)f(X, Y) = 0$ and $\Lambda$ is a division algebra we have $f(x, y) = 0$ for all elements $x, y \in \Lambda$, but then $f \otimes \mathbb{C}(u,v) \neq 0$ as $\Lambda \otimes_{\mathbb{C}(u^n, v^n)} \mathbb{C}(u,v) \simeq M_n(\mathbb{C}(u,v))$, a contradiction finishing the proof. $\hfill \Box$

**Lemma 6** There is a division algebra $\Delta_n$ with center $\mathbb{C}(X_n)^{PGL_n}$ such that

$$\Delta_n \otimes_{\mathbb{C}(X_n)^{PGL_n}} \mathbb{C}(X_n) \simeq M_n(\mathbb{C}(X_n))$$

**Proof:** Let $T_n$ be the subalgebra of $M_n(\mathbb{C}(X_n))$ obtained by adjoining to $\mathbb{G}_n$ all coefficients of characteristic polynomials of its elements. The above argument can be repeated to give that $T_n$ is a domain (because the map $\mathbb{G}_n \to \Lambda$ extends to $T_n \to \Lambda$). Now invert all these coefficients (which are central) then we claim that $\Delta_n = T_n \mathbb{C}(X_n)^{PGL_n}$ (use lemma 3) is a division algebra. If $s \in \mathbb{G}_n$ with characteristic polynomial $s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$, then $s^{-1} = -a_n^{-1}(s^{n-1} + a_1 s^{n-2} + \ldots + a_{n-1})$ belongs to $\Delta_n$. The final statement follows from the proof of the foregoing lemma. $\hfill \Box$
Next, define another division algebra which is of infinite dimension over its center \( \mathbb{C}(X_n)^{PGL_n} \). Let \( \mathfrak{g}_n \) be the semi-direct product Lie algebra \( M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathfrak{sl}_n \). The Lie-bracket is defined by the rule
\[
[(A, B, h), (A', B', h')] = (hA' - A'h - h'A + Ah', hB' - B'h - h'B + Bh', hh' - h'h)
\]
and consider the enveloping algebra \( U(\mathfrak{g}_n) \).

Clearly, \( M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus 0 \) is a commutative sub-Lie algebra of \( \mathfrak{g}_n \) and we can invert all its elements to obtain an intermediate ring
\[
U(\mathfrak{g}_n) \subset \mathbb{C}(X_n)\# U(\mathfrak{sl}_n) \subset D(\mathfrak{g}_n)
\]
where \( D(\mathfrak{g}_n) \) is the division ring of fractions of the Noetherian domain \( U(\mathfrak{g}_n) \) and the intermediate algebra is the \( \mathbb{C} \)- vectorspace \( \mathbb{C}(X_n) \otimes \mathbb{C} U(\mathfrak{sl}_n) \) with multiplication defined by
\[
(f \# h)(f' \# h') = f(h, f') \# hh'
\]
where \( h, f \) is the action by derivations of \( \mathfrak{sl}_n \) (and hence of its enveloping algebra) on \( \mathbb{C}(X_n) \).

We now want to have another interpretation of this intermediate ring. Let \( \mathcal{D}_n \) be the ring of differential operators of the field-extension \( \mathbb{C}(X_n)^{PGL_n} \subset \mathbb{C}(X_n) \). That is, \( \mathcal{D}_n \) is the subalgebra of the endomorphisms of \( \mathbb{C}(X_n) \) generated by \( \mathbb{C}(X_n) \) and the vectorspace \( Der_{\mathbb{C}(X_n)^{PGL_n}}(\mathbb{C}(X_n)) \) of \( \mathbb{C}(X_n)^{PGL_n} \)-derivations on \( \mathbb{C}(X_n) \).

Observe that this is a finite dimensional vectorspace of dimension
\[
trdeg_{\mathbb{C}(X_n)^{PGL_n}}(\mathbb{C}(X_n)) = trdeg_{\mathbb{C}}(\mathbb{C}(X_n)) - trdeg_{\mathbb{C}}(\mathbb{C}(X_n)^{PGL_n}) = n^2 - 1
\]
For more details on differential operators we refer to [6, Ch. 15].

**Lemma 7** There is a canonical isomorphism
\[
\mathbb{C}(X_n)\# \mathbb{C} U(\mathfrak{sl}_n) \simeq \mathcal{D}_n
\]
In particular, \( D(\mathfrak{g}_n) \) is a division algebra with center \( \mathbb{C}(X_n)^{PGL_n} \).

**Proof** : (compare with [1, Prop. 2.1] ) Since \( PGL_n \) is a connected group, we have that
\[
\mathbb{C}(X_n)^{\mathfrak{sl}_n} = \mathbb{C}(X_n)^{PGL_n}
\]
where the first field is the subfield of \( f \in \mathbb{C}(X_n) \) such that \( h, f = 0 \) for all \( h \) in the Lie-algebra of derivations, so the last statement follows from the first.

There is a canonical morphism
\[
\mathbb{C}(X_n)\# \mathbb{C} U(\mathfrak{sl}_n) \to \mathcal{D}_n
\]
Both sides can be filtered, the left-hand by the Poincaré-Birkhoff-Witt filtration on \( U(\mathfrak{sl}_n) \), the right-hand by the order of differential operators. The canonical map is filtration preserving and induces an algebra-morphism between the associated graded rings defined by the map on the degree one parts which is
\[
\mathbb{C}(X_n) \otimes \mathbb{C} \mathfrak{sl}_n \to Der_{\mathbb{C}(X_n)^{PGL_n}}(\mathbb{C}(X_n))
\]
The first statement will follow if we show that this map is an isomorphism. Both sides are \( \mathbb{C}(X_n) \)-vectorspaces of dimension \( n^2 - 1 \) so it suffices to show injectivity which follows by composing with the canonical inclusion \( Der_{\mathbb{C}(X_n)^{PGL_n}}(\mathbb{C}(X_n)) \subset Der_{\mathbb{C}}(\mathbb{C}(X_n)) \) and lemma 4. \( \square \)

Summarizing, we have shown :

**Theorem 2** \( \Delta_n \) is a \( \mathfrak{g}_n \)-bad division algebra. Consequently, the Lie algebra \( \mathfrak{g}_n \) is a counter example to the Gel'fand-Kirillov conjecture.
References


