A non-commutative topology on $\text{rep} A$

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Abstract

We extend the Zariski topology on $\text{simp} A$, the set of all simple finite dimensional representations of $A$, to a non-commutative topology (in the sense of Fred Van Oystaeyen) on $\text{rep} A$, the set of all finite dimensional representations of $A$, using Jordan-Hölder filtrations. The non-commutativity of the topology is enforced by the order of the composition factors.

All algebras will be affine associative $k$-algebras with unit over an algebraically closed field $k$. The *non-commutative affine 'scheme’* associated to an algebra $A$ is, as a set, the disjoint union

$$\text{rep} A = \bigsqcup_n \text{rep}_n A$$

where $\text{rep}_n A$ is the (commutative) affine scheme of $n$-dimensional representations of $A$. In this note we will equip $\text{rep} A$ with a non-commutative topology in the sense of Fred Van Oystaeyen [5 §7.2] (or, more precisely, a slight generalization of it).

Here is the main idea. The twosided prime ideal spectrum $\text{spec} A$ is an (ordinary) topological space via the Zariski topology, see for example [4] or [11 §II.6]. Hence, the subset $\text{simp} A$ of all simple finite dimensional $A$-representations can be equipped with the induced topology. This topology can then be extended to a non-commutative topology on $\text{rep} A$ using Jordan-Hölder filtrations. The non-commutative nature of the topology is enforced by the order of the composition factors.

We give a few examples, connect this notion with that of Reineke’s composition monoid and remark on the difference between quotient varieties and moduli spaces from the perspective of non-commutative topology. Finally, we note that this construction can be generalized verbatim to any Artinian Abelian category as soon as we have a topology on the set of simple objects.
1 The Zariski topology on $\text{simp} \ A$.

Recall that a prime ideal $P$ of $A$ is a twosided ideal satisfying the property that if $I, J \subset P$ then $I \subset P$ or $J \subset P$ for any pair of twosided ideals $I, J$ of $A$. The prime spectrum $\text{spec} \ A$ is the set of all twosided prime ideals of $A$. The Zariski topology on $\text{spec} \ A$ has as its closed subsets

$$\forall(S) = \{P \in \text{spec} \ A \mid S \subset P\}$$

where $S$ varies over all subsets of $A$, see for example [1 Prop. II.6.2]. Note that an algebra morphism $\phi : A \longrightarrow B$ does not necessarily induce a continuous map $\phi^* : \text{spec} \ B \longrightarrow \text{spec} \ A$ but is does so in the case $\phi$ is a central extension in the sense of [1 §II.6].

If $M \in \text{rep}_n A$ is a simple $n$-dimensional representation, there is a defining epimorphism $\psi_M : A \longrightarrow M_n(k)$ and the kernel of this morphism $\ker \psi_M$ is a twosided maximal (hence prime) ideal of $A$. We define the Zariski topology on the set of all simple finite dimensional representations $\text{simp} \ A$ by taking as its closed subsets

$$\forall(S) = \{M \in \text{simp} \ A \mid S \subset \ker \psi_M\}$$

Again, one should be careful that whereas an algebra map $\phi : A \longrightarrow B$ induces a map $\phi^* : \text{rep} \ B \longrightarrow \text{rep} \ A$ it does not in general map $\text{simp} \ B$ to $\text{simp} \ A$ (unless $\phi$ is a central extension).

With $\mathcal{L}_A$ we will denote the set of all open subsets of $\text{simp} \ A$. $\mathcal{L}_A$ will be the set of letters on which to base our non-commutative topology.

2 Non-commutative topologies (and generalizations).

In [5, Chp. 7] Fred Van Oystaeyen defined non-commutative topologies which are generalizations of usual topologies in which it is no longer true that $A \cap A$ is equal to $A$ for an open set $A$. In order to keep dichotomies of possible definitions to a minimum he imposed left-right symmetric conditions on the definition. However, for applications to representation theory it seems that the most natural non-commutative topologies are truly one-sided. For this reason we take some time to generalize some definitions and results of [5 Chp. 7].

We fix a partially ordered set $(\Lambda, \leq)$ with a unique minimal element $0$ and a unique maximal element $1$, equipped with two operations $\land$ and $\lor$. With $i_\Lambda$ we will denote the set of all idempotent elements of $\Lambda$, that is, those $x \in \Lambda$ such that $x \land x = x$. A finite global cover is a finite subset $\{\lambda_1, \ldots, \lambda_n\}$ such that $1 = \lambda_1 \lor \ldots \lor \lambda_n$. In the table below we have listed the conditions for a (one-sided) non-commutative topology. Note that some requirements are less essential than others. For example, the covering condition (A10) is only needed if we want to fit non-commutative topologies in the framework of non-commutative Grothendieck topologies [5] and the weak modularity condition (A9) is not required if every basic open is $\lor$-idempotent (as is the case in most examples).
\[
(x \lor y) \land \cdots \land (x \lor y) = x
\]
\[
(x \lor y) \land \cdots \land (x \lor y) = x
\] (01)

\[
x \lor (q \land p) \supseteq (x \lor q) \land x
\]
\[
x \lor (q \land p) \supseteq (x \lor q) \land p
\] (6)

\[
z \land x \leq x \land z \iff z \leq x
\]
\[
z \land x \leq z \land x \iff z \leq x
\] (8)

\[
z \land x \land z = (z \land x) \land z = z \land (x \land z)
\]
\[
z \land x \land z = (z \land x) \land z = z \land (x \land z)
\] (7)

\[
x = x \land 0
\]
\[
x = x \land 0
\]
\[
x = 0 \land x
\]
\[
I = I \land x
\] (9)

\[
z \land k \leq x \land z \iff z \leq x
\]
\[
z \land k \leq z \land x \iff z \leq x
\] (5)

\[
z \lor (x \lor z) = (z \lor x) \lor z = z \lor (x \lor z)
\]
\[
z \lor (x \lor z) = (z \lor x) \lor z = z \lor (x \lor z)
\] (6)

\[
0 = x \lor 0
\]
\[
0 = x \lor 0
\]
\[
x = x \lor 1
\]
\[
x = x \lor 1
\] (2)

\[
k \leq k \lor x
\]
\[
k \leq k \lor x
\]
\[
x \leq k \lor x
\] (1)
Λ is said to be a right non-commutative topology if and only if the middle and right column conditions of (A1)-(A10) are valid for all \(x, y, z \in \Lambda\), all \(a, b \in i\Lambda\) with \(a \leq b\) and all finite global covers \(\{\lambda_1, \ldots, \lambda_n\}\).

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There are at least two ways of building a genuine non-commutative topology out of these sets of basic opens. We briefly sketch the procedures here and refer to the forthcoming monograph [6] for details in the symmetric case (the one-sided versions present no real problems).

Let \(T(\Lambda)\) be the set of all finite \((\land, \lor)\)-words in the contractible idempotent elements \(i\Lambda\) (that is, \(\lambda \in i\Lambda\) such that for all \(\lambda_1, \lambda_2\) with \(\lambda \leq \lambda_1 \lor \lambda_2\) we have that \(\lambda = (\lambda \land \lambda_1) \lor (\lambda \land \lambda_2)\)). If \(\Lambda\) is a (left,right) non-commutative topology, then so is \(T(\Lambda)\). The \(\lor\)-complete topology of virtual opens \(T'(\Lambda)\) is then the set of all \((\land, \lor)\)-words in the contractible idempotents of finite length in \(\land\) (but not necessarily of finite length in \(\lor\)). This non-commutative topology has properties very similar to that of an ordinary topology and, in fact, has associated to it a commutative shadow.

The second construction, leading to the pattern topology, starts with the equivalence classes of directed systems \(S \subset \Lambda\) (that is, if for all \(x, y \in S\) there is a \(z \in S\) such that \(z \leq x \) and \(z \leq y\)) and where the equivalence relation \(\sim\) is defined by

\[
\forall a \in S, \exists a' \in S, a' \leq a \text{ and } b \leq a' \leq b' \text{ for some } b, b' \in S'. \\
\forall b \in S', \exists b' \in S', b' \leq b \text{ and } a \leq b' \leq a' \text{ for some } a, a' \in S.
\]

One can extend the \(\land, \lor\) operations on \(\Lambda\) to the equivalence classes \(C(\Lambda) = \{[S] \mid S \text{ directed}\}\) in the obvious way such that also \(C(\Lambda)\) is a (left,right) non-commutative topology. A directed set \(S \subset \Lambda\) is said to be idempotent if for all \(a \in S\), there is an \(a' \in S \cap i\Lambda\) such that \(a' \leq a\). If \(S\) is idempotent then \([S] \in iC(\Lambda)\) and those idempotents will be called strong idempotents. The pattern topology \(\Pi(\Lambda)\) is the (left,right) non-commutative topology of finite \((\land, \lor)\)-words in the strong idempotents of \(C(\Lambda)\). A directed system \([S]\) is called a point iff \([S] \leq \lor[S_\alpha]\) implies that \([S] \leq [S_\alpha]\) for some \(\alpha\).

### 3 The basic opens.

For an \(n\)-dimensional representation \(M\) of \(A\) we call a finite filtration of length \(u\)

\[
\mathcal{F}^u : 0 = M_0 \subset M_1 \subset \ldots \subset M_u = M
\]

of \(A\)-representations a Jordan-Hölder filtration if the successive quotients

\[
\mathcal{F}_i = \frac{M_i}{M_{i-1}}
\]
are simple \( A \)-representations. Recall that \( \mathcal{L}_A \) is the set of all open subsets \( V \) of \( \text{simp} \ A \). With \( \mathbb{W}_A \) we denote the non-commutative words in these letters
\[
\mathbb{W}_A = \{ V_1 \ldots V_k \mid V_i \in \mathcal{L}_A, k \in \mathbb{N} \}
\]
For a given word \( w = V_1 V_2 \ldots V_k \in \mathbb{W}_A \) we define the left basic open set
\[
\mathcal{O}^l_w = \{ M \in \text{rep} \ A \mid \exists \mathcal{F}^w \text{ Jordan-Hölder filtration on } M \text{ such that } \mathcal{F}_i \in V_i \}
\]
and the right basic open set
\[
\mathcal{O}^r_w = \{ M \in \text{rep} \ A \mid \exists \mathcal{F}^w \text{ Jordan-Hölder filtration on } M \text{ such that } \mathcal{F}_{w-i} \in V_{k-i} \}
\]
Finally, to make these definitions symmetric we define the basic open set
\[
\mathcal{O}_w = \{ M \in \text{rep} \ A \mid \exists \mathcal{F}^w \text{ Jordan-Hölder filtration on } M \text{ such that } \mathcal{F}_{i_{j-1}} \in V_{j-1} \}
\]
Clearly, \( \mathcal{O}^l_w \) consists of those representations having prescribed bottom structure, whereas \( \mathcal{O}^r_w \) consists of those with prescribed top structure. In order to avoid three sets of definitions we will denote from now on \( \mathcal{O}_\bullet \) whenever we mean \( \bullet \in \{ l, r, \emptyset \} \).

If \( w = L_1 \ldots L_k \) and \( w' = M_1 \ldots M_l \), we will denote with \( w \cup w' \) the multi-set \( \{ N_1, \ldots, N_m \} \) where each \( N_i \) is one of \( L_j, M_j \) and \( N_i \) occurs in \( w \cup w' \) as many times as its maximum number of factors in \( w \) or \( w' \). With \( \text{rep}(w \cup w') \) we denote the subset of \( \text{rep} \ A \) consisting of the representations of \( M \) having a Jordan-Hölder filtration having factor-multi-set containing \( w \cup w' \). For any triple of words \( w, w' \) and \( w'' \) we denote \( \mathcal{O}_{w''}(w \cup w') = \mathcal{O}_{w''} \cap \text{rep}(w \cup w') \).

We define an equivalence relation on the basic open sets by
\[
\mathcal{O}_w \approx \mathcal{O}_{w'} \iff \mathcal{O}_w(w \cup w') = \mathcal{O}_{w'}(w \cup w')
\]
The reason for this definition is that the condition of \( M \in \mathcal{O}_w \) is void if \( M \) does not have enough Jordan-Hölder components to get all factors of \( w \) which makes it impossible to define equality of basic open sets defined by different words.

We can now define the partially ordered sets \( \Lambda^*_A \) as consisting of all basic open subsets \( \mathcal{O}_w \) of \( \text{rep} \ A \). The partial ordering \( \leq \) is induced by set-theoretic inclusion modulo equivalence, that is,
\[
\mathcal{O}_w \leq \mathcal{O}_{w'} \iff \mathcal{O}_w(w \cup w') \subseteq \mathcal{O}_{w'}(w \cup w')
\]
As a consequence, equality \( = \) in the set \( \Lambda^*_A \) coincides with equivalence \( \approx \). Observe that these partially ordered sets have a unique minimal and a unique maximal element (upto equivalence)
\[
0 = \emptyset = \mathcal{O}_0^* \quad \text{and} \quad 1 = \text{rep} \ A = \mathcal{O}_{\text{simp} \ A}^*
\]
The operations \( \vee \) and \( \wedge \) are defined as follows : \( \vee \) is induced by ordinary set-theoretic union and \( \wedge \) is induced by concatenation of words, that is
\[
\mathcal{O}_w \wedge \mathcal{O}_{w'} \approx \mathcal{O}_{ww'}
\]
Theorem 1 With notations as before :

- \((\Lambda^l_A, \leq, \preceq, 0, 1, \lor, \land)\) is a left non-commutative topology on \(\text{rep} \ A\).
- \((\Lambda^r_A, \leq, \preceq, 0, 1, \lor, \land)\) is a right non-commutative topology on \(\text{rep} \ A\).

Proof. The tedious verification is left to the reader. Here, we only stress the importance of the equivalence relation for example in verifying \(x \land 1 = x\). So, let \(w = L_1 \ldots L_k\)

\[O^l_w \land 1 = O^l_{L_1 \ldots L_k \mathcal{S} \mathcal{I} \mathcal{M} \mathcal{A}} \subset O^l_w\]

and this inclusion is proper (look at elements in \(O^l_w\) having exactly \(k\) composition factors). However, as soon as the representation has \(k + 1\) composition factors, it is contained in the left hand side whence \(O^l_w \land 1 \approx O^l_w\). A similar argument is needed in the covering condition. □

Note however that \((\Lambda_A, \leq, \preceq, 0, 1, \lor, \land)\) is not necessarily a non-commutative topology: the problematic conditions are \(O^l_w \land 1 = O^l_w = 1 \land O^l_w\) and the covering condition. The reason is that for \(w = L_1 \ldots L_k\) as before and \(M \in O^l_w\) having \(> k\) factors, it may happen that the last factor is the one in \(L_k\) leaving no room for a successive factor in \(\mathcal{S} \mathcal{I} \mathcal{M} \mathcal{A}\) (whence \(O^l_w \land 1\) is not equivalent to \(O^l_w\)).

Example 1 Let \(A\) be a finite dimensional algebra, then \(A\) has a finite number of simple representations \(\mathcal{S} \mathcal{I} \mathcal{M} \mathcal{A} = \{S_1, \ldots, S_n\}\) and the Zariski topology is the discrete topology. If for some \(1 \leq i, j \leq n\) we have that

\[\text{Ext}^1_A(S_i, S_j) = 0 \quad \text{and} \quad \text{Ext}^1_A(S_j, S_i) \neq 0\]

then \(\Lambda^l_A\) is a genuinely non-commutative topology, for example

\[O^l_{S_i} \land O^l_{S_j} = O^l_{S_i S_j} \neq O^l_{S_j S_i} = O^l_{S_j} \land O^l_{S_i}\]

as a non-trivial extension \[\begin{array}{c}
0 \rightarrow S_i \rightarrow X \rightarrow S_j \rightarrow 0
\end{array}\] belongs to \(O^l_{S_i, S_j}(S_i S_j \cup S_j S_i)\) but not to \(O^l_{S_j, S_i}(S_i S_j \cup S_j S_i)\).

4 Reineke’s mon(str)oid.

When \(A\) is the path algebra of a quiver without oriented cycles we can generalize the foregoing example and connect the previous definitions to the composition monoid introduced and studied by Markus Reineke in [2].

Let \(Q\) be a quiver without oriented cycles, then its path algebra \(A = \mathbb{k}Q\) is finite dimensional hereditary with all simple representations one-dimensional and in one-to-one correspondence with the vertices of \(Q\). For every dimension \(n\) we have that

\[\text{rep}_n A = \bigsqcup_{|\alpha| = n} GL_n \times^{GL(\alpha)} \text{rep}_\alpha Q\]
where $\alpha$ runs over all dimension vectors of total dimension $n$ and where $\text{rep}_{\alpha} Q$ is the affine space of all $\alpha$-dimensional representations of the quiver $Q$ with base-change group action by $GL(\alpha)$.

The Reineke monstroid $\mathcal{M}(Q)$ has as its elements the set of all irreducible closed $GL(\alpha)$-stable subvarieties of $\text{rep}_{\alpha} Q$ for all dimension vectors $\alpha$, equipped with a product $A \ast B = \{ X \in \text{rep}_{\alpha+\beta} Q \mid$ there is an exact sequence $0 \rightarrow M \rightarrow X \rightarrow N \rightarrow 0 \rightarrow M \in A, N \in B \}$ if $A$ (resp. $B$) is an element of $\mathcal{M}(Q)$ contained in $\text{rep}_{\alpha} Q$ (resp. in $\text{rep}_{\beta} Q$). It is proved in [2, lemma 2.2] that $A \ast B$ is again an element of $\mathcal{M}(Q)$. This defines a monoid structure on $\mathcal{M}(Q)$ which is too unwieldy to study directly. Observe that we changed the order of the terms wrt. the definition given in [2]. That is, we will work with the opposite monoid of [2].

On the other hand, the Reineke composition monoid is very tractable. It is the submonoid $C(Q)$ of $\mathcal{M}(Q)$ generated by the vertex-representation spaces $R_i = \text{rep}_{\delta_i} Q$. These generators satisfy specific commutation relations which can be read off from the quiver structure, see [2, §5]. For example, if there are no arrows between $v_i$ and $v_j$ then

$$R_i \ast R_j = R_j \ast R_i$$

and if there are no arrows from $v_i$ to $v_j$ but $n$ arrows from $v_j$ to $v_i$, then

$$\begin{cases} R_i^{(n+1)} \ast R_j = R_i^n \ast R_j \ast R_i \\ R_i \ast R_j^{(n+1)} = R_j \ast R_i \ast R_j^n \end{cases}$$

For more details on the structure of $C(Q)$ we refer to [2, §5].

There is a relation between $C(Q)$ and the left- and right- non-commutative topologies $\Lambda^l_A$ and $\Lambda^r_A$. Because the Zariski topology on $\text{simp } A$ is the discrete topology on the set $\{S_1, \ldots, S_k\}$ of vertex simples, it is important to understand $O^r_w$ where $w$ is a word in the $S_i$, say $w = S_{i_1}S_{i_2} \ldots S_{i_u}$. In fact, we could have based our definition of a one-sided non-commutative topology on the set $\mathcal{L}_A$ of irreducible open subsets of $\text{simp } A$ and then these basic opens would be all. If $C$ is a $GL(\alpha)$-stable subset of $\text{rep}_{\alpha} Q$ with $|\alpha| = n$, we will denote the subset $GL_n \times^{GL(\alpha)} C$ of $\text{rep}_n A$ by $\tilde{C}$.

**Proposition 1**

$$O^l_w = \bigcup_{w'} \tilde{A}_{w'} \quad \text{resp.} \quad O^r_w = \bigcup_{w'} \tilde{A}_{w'}$$

where $A_{w'}$ is a *-word in the generators $R_i$ of the composition monoid such that $w'$ can be rewritten (using the relations in $C(Q)$) in the form

$$w' = R_{i_1} \ast R_{i_2} \ast \ldots \ast R_{i_u} \ast w'' \quad \text{resp.} \quad w' = w'' \ast R_{i_1} \ast R_{i_2} \ast \ldots \ast R_{i_u}$$

for another *-word $w''$.  


Also, the equivalence relation introduced before can be expressed in terms of $C(Q)$. If $w = S_{i_1}S_{i_2}...S_{i_u}$ and $w' = S_{j_1}S_{j_2}...S_{j_v}$ such that $w \cup w' = \{S_{k_1}, ..., S_{k_w}\}$, then

**Proposition 2** $O_{w}^{l} \approx O_{w'}^{l}$ if and only if every $\ast$-word $v = R_{a_1} \ast \ldots \ast R_{a_z}$ containing in it distinct factors $R_{k_1}, \ldots, R_{k_w}$ which can be brought in $C(Q)$ in the form

$$v = R_{i_1} \ast \ldots \ast R_{i_u} \ast v'$$

can also be written in the form

$$v = R_{j_1} \ast \ldots \ast R_{j_v} \ast v''$$

(and conversely). A similar result describes $O_{w}^{r} \approx O_{w'}^{r}$.

In particular, in this setting there will be hardly any idempotent basic opens (that is, satisfying $O_{w}^{r} \land O_{w'}^{r} \approx O_{w'}^{r}$). Clearly, if $\{S_{e_1}, \ldots, S_{e_a}\}$ are simples such that the quiver restricted to $\{v_{e_1}, \ldots, v_{e_a}\}$ has no arrows, then any word $w$ in the $S_{e_j}$ gives an idempotent $O_{w}^{r}$. In the following section we will give an example where every basic open is idempotent and hence we get a commutative topology.

## 5 The commutative case.

If $A$ is a commutative affine $k$-algebra, then any simple representation is one-dimensional, $\mathbf{simp} A = X_{A}$ the affine (commutative) variety corresponding to $A$ and the Zariski topologies on both sets coincide. Still, one can define the non-commutative topologies on $\mathbf{rep} A$. However,

**Proposition 3** If $A$ is a commutative affine $k$-algebra, then both $\Lambda_{A}^{l}$ and $\Lambda_{A}^{r}$ are commutative topologies. That is, for all words $w$ and $w'$ in $L_{A}$ we have

$$O_{w}^{l} \land O_{w'}^{l} \approx O_{w'}^{l} \land O_{w}^{l} \quad \text{and} \quad O_{w}^{r} \land O_{w'}^{r} \approx O_{w'}^{r} \land O_{w}^{r}$$

**Proof.** We claim that every basic open $O_{w}^{l}$ is idempotent. Observe that all simple $A$-representations are one-dimensional and that there are only self-extensions of those, that is, if $S$ and $T$ are non-isomorphic simples, then $\text{Ext}_{A}^{1}(S, T) = 0 = \text{Ext}_{A}^{1}(T, S)$. However, there are self-extensions with the dimension of $\text{Ext}_{A}^{1}(S, S)$ being equal to the dimension of the tangent space at $X_{A}$ in the point corresponding to $S$. As a consequence we have for any Zariski open subsets $U$ and $V$ of $X_{A}$ that

$$O_{U \cap V}^{l} = O_{U}^{l} \cap O_{V}^{l}$$

as we can change the order of the filtration factors (a representation $M$ is the direct sum of submodules $M_{1} \oplus \ldots \oplus M_{s}$ with each $M_{i}$ concentrated in a single simple $S_{i}$ and we can add the successive $S_{i}$ factors of $M$ at any wanted place in the filtration sequence). Hence, for every word $w$ we have that

$$O_{w}^{l} \approx O_{w}^{l} \land O_{w}^{l}$$
and also for any pair of words $w$ and $w'$ we have that
\[ O^l_w \land O^l_{w'} = O^l_{ww'} = O^l_{w'w} = O^l_{w'} \land O^l_w. \]

Observe that in [5] it is proved that a non-commutative topology in which every basic open is idempotent is commutative. We cannot use this here as the proof of that result uses both the left- and right- conditions. However, we are dealing here with a very simple example.

\[ \Box \]

6 Quotient varieties versus moduli spaces.

Having defined a one-sided non-commutative topology on $\text{rep} A$ we can ask about the induced topology on the quotient variety $\mathcal{i}ss A$ of all isomorphism classes of semi-simple $A$-representations or on the moduli space $\text{moduli}_\theta A$ with respect to a certain stability structure $\theta$, cfr. [3]. Experience tells us that it is a lot easier to work with quotient varieties than with moduli spaces and non-commutative topology may give a partial explanation for this.

Indeed, as the points of $\mathcal{i}ss A$ are semi-simple representations, it is clear that the induced non-commutative topology on $\mathcal{i}ss A$ is in fact commutative. However, as the points of $\text{moduli}_\theta A$ correspond to isomorphism classes of direct sums of stable representations (not simples!), the induced non-commutative topology on $\text{moduli}_\theta A$ will in general remain non-commutative. Still, in nice examples, such as representations of quivers, one can define another non-commutative topology on $\text{moduli}_\theta A$ which does become commutative. Use universal localization to cover $\text{moduli}_\theta A$ by opens isomorphic to $\mathcal{i}ss A_{\Sigma}$ for some families $\Sigma$ of maps between projectives and equip $\text{moduli}_\theta A$ with a non-commutative topology (which then will be commutative!) obtained by gluing the induced non-commutative topologies on the $\text{rep} A_{\Sigma}$.

7 Generalizations.

It should be evident that our construction can be carried out verbatim in the setting of any Artinian Abelian category (that is, an Abelian category having Jordan-Hölder sequences) as soon as we have a natural topology on the set of simple objects. In fact, the same procedure can be applied when we have a left (or right) non-commutative topology on the simples.

In fact, the construction may even be useful in Abelian categories in which every object is filtered by special objects on which we can define a (one-sided) (non-commutative) topology.

References

[1] Claudio Procesi, Rings with polynomial identities, Marcel Dekker (1973)


