BRAID GROUP $B_3$ IRREDUCIBLES
- A DIY GUIDE -

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ABSTRACT. This note tells you how to construct a $k(n)$-dimensional family of (isomorphism classes of) irreducible representations of dimension $n$ for the three string braid group $B_3$, where $k(n)$ is an admissible function of your choosing; for example take $k(n) = \lfloor \frac{n}{2} \rfloor + 1$ as in [2] and [3].

(step 1) Learn the basics. The three string braid group $B_3$ is the group $\langle \sigma_1, \sigma_2 | \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$ and its center is cyclic with generator $c = (\sigma_1 \sigma_2)^3 = (\sigma_1 \sigma_2 \sigma_1)^2$. The quotient group

$B_3/\langle c \rangle = \langle u, v | u^2 = v^3 = e \rangle \cong C_2 \ast C_3 \cong \Gamma_0$

is the modular group $PSL_2(\mathbb{Z})$ where $u$ and $v$ are the images of $\sigma_1 \sigma_2$ resp. $\sigma_1 \sigma_2 \sigma_1$.

By Schur’s lemma, the central element $c$ acts as $\lambda I_n$ (where $\lambda \in \mathbb{C}^\times$) on any $n$-dimensional irreducible $B_3$-representation. Hence, it is enough to construct a $k(n)-1$-dimensional family of $n$-dimensional irreducible representations of the modular group $\Gamma_0$.

If $V$ is an $n$-dimensional $\Gamma_0$ representation, we can decompose it into eigenspaces for the action of $C_2 = \langle u \rangle$ and $C_3 = \langle v \rangle$:

$V_1 \oplus V_2 = V \downarrow_{C_2} = V \downarrow_{C_3} = W_1 \oplus W_2 \oplus W_3$

If the dimension of $V_i$ is $a_i$ and that of $W_j$ is $b_j$, we say that $V$ is a $\Gamma_0$-representation of dimension vector $\alpha = (a_1, a_2; b_1, b_2, b_3)$. Choosing a basis $B_1$ of $V$ wrt. the decomposition $V_1 \oplus V_2$ and a basis $B_2$ wrt. $W_1 \oplus W_2 \oplus W_3$, we can view the basechange matrix $B_1 \longrightarrow B_2$ as an $\alpha$-dimensional representation $V_Q$ of the quiver $Q$.

Bruce Westbury [6] has shown that $V$ is an irreducible $\Gamma_0$-representation if and only if $V_Q$ is a $\theta$-stable $Q$-representation where $\theta = (-1, -1; 1, 1, 1)$ and that the two notions of isomorphism coincide. The Euler-form $\chi_Q$ of the quiver $Q$ is the bilinear form on $\mathbb{Z}^{\oplus 5}$ determined by the matrix

$$
\begin{bmatrix}
1 & 0 & -1 & -1 & -1 \\
0 & 1 & -1 & -1 & -1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Westbury also showed that if there exists a $\theta$-stable $\alpha$-dimensional $Q$-representation, then there is an $1 - \chi_Q(\alpha, \alpha)$ dimensional family of isomorphism classes of such representations (and a Zariski open subset of them will correspond to isomorphism classes of irreducible $\Gamma_0$-representations). Hence, an admissible function $k(n)$ is one such that for all $n$ we have $k(n) \leq 2 - \chi_Q(\alpha_n, \alpha_n)$ for a dimension vector $\alpha_n = (a_1, a_2; b_1, b_2, b_3)$ such that $n = a_1 + a_2$ and there exists a $\theta$-stable $\alpha_n$-dimensional $Q$-representation. Note that Aidan Schofield [5] gave an inductive procedure to determine the dimension vectors of stable representations.
(step 2) Choose known non-isomorphic $\Gamma_0$-irreducibles and their corresponding $\theta$-stable $Q$-representations $\{V_i : i \in I\}$. Here are some obvious choices: using the foregoing and standard quiverology, there are 6 irreducible 1-dimensional $\Gamma_0$-representations $S_{ij}$ and there are 3 one-parameter families of 2-dimensional simple $\Gamma_0$-representations $T_i(\lambda)$. Below the corresponding $Q$-representations for $S_{21}$ and $T_2(\lambda)$ (the other cases are similar)

$$S_{21} = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 2 & 1 \\
\end{array} \quad T_2(\lambda) = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 2 & 1 & 1 \\
\end{array}$$

More interesting choices are the $Q$-representations corresponding to irreducible continuous representations of $\Gamma_0$, the profinite completion of the modular group. For example, a simple factor of the monodromy representation associated to a dessin d’enfant or an irreducible representation of a finite group generated by an order two and an order three element, for example the monster group $M$. Pick your favourite collection of non-isomorphic $\{V_i\}$.

(step 3) Compute the local quiver of the collection $\{V_i : i \in I\}$ as in e.g. [1]. That is, we make a new quiver $\Delta$ having one vertex $v_i$ for every $V_i$. If $\alpha_i$ is the dimension vector of the $\theta$-stable $Q$-representation determined by $V_i$, then there are $1 - \chi_\alpha(\alpha_i, \alpha_i)$ loops in vertex $v_i$ in $\Delta$ and there are exactly $-\chi_\alpha(\alpha_i, \alpha_j)$ oriented arrows starting in vertex $v_i$ and ending in vertex $v_j$ in $\Delta$.

For each $n \in \mathbb{N}$ take a finite subquiver $\Delta_n$ of $\Delta$ (say, on the vertices $\{v_{n,1}, \ldots, v_{n,k}\}$) then [1] asserts that there is an étale map between a Zariski open subset of the moduli space $M^\alpha_n(Q, \theta)$ of $\theta$-semi-stable $Q$-representations of dimension vector $\alpha = \alpha_{n,1} + \alpha_{n,2} + \ldots + \alpha_{n,k}$ around the $Q$-representation $V_{n,1} \oplus V_{n,2} \oplus \ldots \oplus V_{n,k}$ and the moduli space of semi-simple $\Delta_n$-representations of dimension vector $1 = (1,1,\ldots,1)$ around the zero-representation. Moreover, in this étale correspondence, (isomorphism classes of) simple $\Delta_n$-representations correspond to (isomorphism classes of) $\theta$-stable representations.

By the results from [4] we have accomplished our objective, provided we can find for each $n$ a subquiver $\Sigma_n$ of $\Delta_n$ satisfying the following conditions

- $\Sigma_n$ is strongly connected, meaning that any two vertices are connected via an oriented circuit in $\Sigma_n$, and
- $1 - \chi_{\Sigma_n}(1,1) = k(n) - 1$ where $\chi_{\Sigma_n}$ is the Euler-form (as above) of the quiver $\Sigma_n$.

An example: consider the set $\{V_0 = S_{11}, V_1 = T_1(\lambda_1), V_2 = T_2(\lambda_2), V_3 = T_1(\lambda_3), V_4 = T_2(\lambda_4), V_5 = T_1(\lambda_5), \ldots\}$ with $\lambda_i \neq \lambda_j$ if $i \neq j$. Then, the quiver $\Delta$ has exactly one loop in each vertex $v_i$ (except in $v_0$) and exactly one arrow $v_i \rightarrow v_j$ whenever $i \neq j$ mod 2. Let $\Delta_n$ be the full subquiver on the first $\lfloor \frac{n}{2} \rfloor$ vertices and $\Sigma_n$ the subquiver below (on vertices $\{v_{11}, \ldots, v_{\lfloor \frac{n}{2} \rfloor}\}$ if $n$ is even and on $\{v_{0}, v_{1}, \ldots, v_{\lfloor \frac{n}{2} \rfloor}\}$ if $n$ is odd). Then, the indicated representations give an $\lfloor \frac{n}{2} \rfloor$-parameter family of simple $\Sigma_n$ (and hence also $\Delta_n$)-representations

- $(n$ even$)$: \begin{align*}
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 & 1 \\
\end{array} & \begin{array}{cccc}
1 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 2 & 2 & 2 \\
\end{array} & \begin{array}{cccc}
1 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 3 & 3 & 3 \\
\end{array} & \begin{array}{cccc}
1 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 4 & 4 & 4 \\
\end{array} & \ldots \\
\end{align*}

- $(n$ odd$)$: \begin{align*}
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 2 & 1 \\
\end{array} & \begin{array}{cccc}
1 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 2 & 3 & 2 \\
\end{array} & \begin{array}{cccc}
1 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 3 & 4 & 3 \\
\end{array} & \begin{array}{cccc}
1 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 4 & 5 & 4 \\
\end{array} & \ldots \\
\end{align*}

Using the étale map these representations give an $\lfloor \frac{n}{2} \rfloor$-parameter family of $\theta$-stable $Q$-representations and hence of irreducible $n$-dimensional $\Gamma_0$-representations, and hence by Schur an $\lfloor \frac{n}{2} \rfloor + 1$-parameter family of isomorphism classes of irreducible $B_3$-representations.
(step 4) **Reverse-engineer** the above general argument to fit your specific example.

**REFERENCES**


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