DENSE FAMILIES OF $B_3$ REPRESENTATIONS AND BRAID REVERSION

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Abstract. We prove that all components of $n$-dimensional simple representations of the three string braid group $B_3$ are densely parametrized by rational quiver varieties and give explicit parametrizations for $n < 12$. As an application we show that there is a unique component of 6-dimensional simple $B_3$-representations detecting braid-reversion.

A knot is said to be invertible if it can be deformed continuously to itself, but with the orientation reversed. There exist non-invertible knots, the unique one with a minimal number of crossings is knot $8_{17}$ which is the closure of the three string braid $b = \sigma_1^{-2} \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2^{-2}$.

Proving knot-invertibility of $8_{17}$ essentially comes down to separating the conjugacy class of the braid $b$ from that of its reversed braid $b' = \sigma_2^2 \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1^{-2}$. Recall that knot invariants derived from quantum groups cannot detect invertibility, see [6]. On the other hand, $\text{Tr}_V(b) \neq \text{Tr}_V(b')$ for a sufficiently large $B_3$-representation $V$.

In fact, Bruce Westbury discovered such 12-dimensional representations, and asked for the minimal dimension of a $B_3$-representation able to detect a braid from its reversed braid, [16].

Imre Tuba and Hans Wenzl have given a complete classification of all simple $B_3$-representations in dimension $\leq 5$, [14]. By inspection, one verifies that none of these representation can detect invertibility, whence the minimal dimension must be 6. Unfortunately, no complete classification is known of simple $B_3$-representations of dimension $\geq 6$. In this note, we propose a general method to solve this and similar separation problems for three string braids.

Let $\text{rep}_n B_3$ be the affine variety of all $n$-dimensional representations of the three string braid group $B_3$. There is a base change action of $GL_n$ on this variety having as its orbits the isomorphism classes of $n$-dimensional representations. The GIT-quotient of this action, that is, the variety classifying closed orbits

$$\text{rep}_n B_3 // GL_n = \text{iss}_n B_3$$

is the affine variety $\text{iss}_n B_3$ whose points correspond to the isomorphism classes of semi-simple $n$-dimensional $B_3$-representations. In general, $\text{iss}_n B_3$ will have
several irreducible components

$$\text{iss}_\alpha B_3 = \bigcup_\alpha \text{iss}_\alpha B_3$$

If we can prove that $Tr_W(b_1) = Tr_W(b_2)$ for all representations $W$ in a Zariski-dense subset $Y_\alpha \subset \text{iss}_\alpha B_3$, then no representation $V$ in that component will be able to separate $b_1$ from $b_2$. In order to facilitate the calculations, we would like to parametrize the dense family $Y_\alpha$ by a minimal number of free parameters. That is, if $\dim \text{iss}_\alpha B_3 = d$ we would like to construct explicitly a morphism $X_\alpha \longrightarrow \text{iss}_\alpha B_3$ from a rational affine variety of dimension $d$, having a Zariski dense image in $\text{iss}_\alpha B_3$.

The theory of Luna-slices in geometric invariant theory, see [9] and [10], will provide us with a supply of affine varieties $X_\alpha$ and specific étale 'action'-maps $X_\alpha \longrightarrow \text{iss}_\alpha B_3$. Rationality results on quiver representations, see [2] and [13], will then allow us to prove rationality of some specific of these varieties $X_\alpha$. As the modular group $\Gamma_0 = B_3/\langle c \rangle$ is a central quotient of $B_3$ it suffices to obtain these results for $\Gamma_0$. In the first two sections we will prove

**Theorem 1.** The affine variety classifying $n$-dimensional semi-simple representations of the modular group decomposes into a disjoint union of irreducible components

$$\text{iss}_\alpha \Gamma_0 = \bigcup_\beta \text{iss}_\beta \Gamma_0$$

There exists a fixed quiver $Q$ having the following property. For every component $\text{iss}_\beta \Gamma_0$ containing a simple representation, there is a $Q$-dimension vector $\alpha$ with rational quotient variety $\text{iss}(Q,\alpha)$ and an étale action map

$$\text{iss}(Q,\alpha) \longrightarrow \text{iss}_\beta \Gamma_0$$

having a Zariski dense image in $\text{iss}_\beta \Gamma_0$.

We apply this general method to solve Westbury’s separation problem. Of the four irreducible components of $\text{iss}_6 B_3$ the three of dimension 6 cannot detect invertibility, whereas the component of dimension 8 can. A specific representation in that component is given by the matrices

$$\sigma_1 = \begin{bmatrix}
\rho + 1 & \rho - 1 & \rho - 1 & \rho - 1 & -\rho + 1 & -\rho + 1 \\
-2\rho - 1 & -1 & -2\rho - 1 & 2\rho + 1 & -2\rho - 1 & 2\rho + 1 \\
\rho + 2 & \rho + 2 & -\rho & \rho - 2 & -\rho - 2 & \rho + 2 \\
-\rho - 2 & -3\rho & \rho + 2 & -\rho + 2 & 3\rho & -\rho - 2 \\
\rho - 1 & -\rho + 1 & 3\rho + 3 & -\rho + 1 & 3\rho + 1 & -3\rho - 3 \\
-3 & -2\rho - 1 & 2\rho + 1 & 3 & 2\rho + 1 & -2\rho - 3
\end{bmatrix}$$

$$\sigma_2 = \begin{bmatrix}
\rho + 1 & \rho - 1 & \rho - 1 & -\rho + 1 & \rho - 1 & -\rho - 1 \\
-2\rho - 1 & -1 & -2\rho - 1 & -2\rho - 1 & 2\rho + 1 & -2\rho - 1 \\
\rho + 2 & \rho + 2 & -\rho & \rho + 2 & \rho + 2 & -\rho - 2 \\
\rho + 2 & 3\rho & -\rho - 2 & \rho + 2 & 3\rho & -\rho - 2 \\
-\rho + 1 & \rho - 1 & -3\rho - 3 & -\rho + 1 & 3\rho + 1 & -3\rho - 3 \\
3 & 2\rho + 1 & -2\rho - 1 & 3 & 2\rho + 1 & -2\rho - 3
\end{bmatrix}$$

where $\rho$ is a primitive third root of unity. One verifies that $Tr(b) = -7128\rho - 1092$ for the braid $b$ describing knot $8_{17}$, whereas $Tr(b') = 7128\rho + 6036$ for the reversed braid $b'$. 

In order to facilitate the application of this method to other separation problems of three string braids, we provide in section three explicit rational parametrizations of dense families of simple \(n\)-dimensional \(B_3\)-representations for all components and all \(n \leq 11\). This can be viewed as a first step towards extending the Tuba-Wenzl classification \([14]\).

1. Luna slices for representations of the modular group

In this section we recall the reduction, due to Bruce Westbury \([15]\), of the study of finite dimensional simple representations of the three string braid group \(B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle\) to those of a particular quiver \(Q_0\). We will then identify the Luna slices at specific \(Q_0\)-representations to representation spaces of corresponding local quivers as introduced and studied in \([1]\).

Recall that the center of \(B_3\) is infinite cyclic with generator \(c = (\sigma_1 \sigma_2)^3 = (\sigma_1 \sigma_2 \sigma_1)^2\) and hence that the corresponding quotient group (taking \(S = \sigma_1 \sigma_2 \sigma_1\) and \(T = \sigma_1 \sigma_2\))

\[
B_3/\langle c \rangle = \langle S, T \mid S^2 = T^3 = e \rangle \simeq C_2 \ast C_3
\]

is the free product of cyclic groups of order two and three and therefore isomorphic to the modular group \(\Gamma_0 = PSL_2(\mathbb{Z})\).

By Schur’s lemma, \(c\) acts via scalar multiplication with \(\lambda \in \mathbb{C}^*\) on any finite dimensional irreducible \(B_3\)-representation, hence it suffices to study the irreducible representations of the modular group \(\Gamma_0\). Bruce Westbury \([15]\) established the following connection between irreducible representations of \(\Gamma_0\) and specific stable representations of the directed quiver \(Q_0\):

\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ
\end{array}
\]

For \(V\) an \(n\)-dimensional representation of \(\Gamma_0\), decompose \(V\) into eigenspaces with respect to the actions of \(S\) and \(T\)

\[
V_+ \oplus V_- = V = V_1 \oplus V_\rho \oplus V_{\rho^2}
\]

(\(\rho\) a primitive 3rd root of unity). Denote the dimensions of these eigenspaces by \(a = \text{dim}(V_+), b = \text{dim}(V_-)\) resp. \(x = \text{dim}(V_1), y = \text{dim}(V_\rho)\) and \(z = \text{dim}(V_{\rho^2})\), then clearly \(a + b = n = x + y + z\).

Choose a vector-space basis for \(V\) compatible with the decomposition \(V_+ \oplus V_-\) and another basis of \(V\) compatible with the decomposition \(V_1 \oplus V_\rho \oplus V_{\rho^2}\), then the associated base-change matrix \(B \in GL_n(\mathbb{C})\) determines a \(Q_0\)-representation \(V_B\) of
A $Q_0$-representation $W$ of dimension vector $\alpha$ is said to be $\theta$-stable, resp. $\theta$-semi-stable if for every proper sub-representations $W'$, with dimension vector $\beta = (a', b'; x', y', z')$, we have that $x' + y' + z' > a' + b'$, resp. $x' + y' + z' \geq a' + b'$.

**Theorem 2** (Westbury, [15]). $V$ is an $n$-dimensional simple $\Gamma_0$-representation if and only if the corresponding $Q_0$-representation $V_B$ is $\theta$-stable. Moreover, $V \simeq W$ as $\Gamma_0$-representations if and only if corresponding $Q_0$-representations $V_B$ and $W_B'$ are isomorphic as quiver-representations.

The affine GIT-quotient $\text{iss}_n \Gamma_0 = \text{rep}_n \Gamma_0/GL_n$ classifying isomorphism classes of $n$-dimensional semi-simple $\Gamma_0$-representations decomposes into a disjoint union of irreducible components

$$\text{iss}_n \Gamma_0 = \bigsqcup_{\alpha} \text{iss}_\alpha \Gamma_0$$

one component for every dimension vector $\alpha = (a, b; x, y, z)$ satisfying $a + b = n = x + y + z$. If $\alpha = (a, b; x, y, z)$ satisfies $x, y, z \neq 0$, then the component $\text{iss}_\alpha \Gamma_0$ contains an open subset of simple representations if and only if $\max(x, y, z) \leq \min(a, b)$. In this case, the dimension of $\text{iss}_\alpha \Gamma_0$ is equal to $1 + n^2 - (a^2 + b^2 + x^2 + y^2 + z^2)$.

The remaining simple $\Gamma_0$-representations (that is, those of dimension vector $\alpha = (a, b; x, y, z)$ with $x, y, z = 0$) are of dimension one or two. There are 6 one-dimensional simples (of the abelianization $\Gamma_{0,ab} = C_2 \times C_3$) corresponding to the $Q_0$-representations

$$\begin{align*}
S_1 &= \begin{array}{c}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & 1 & 1 \\
1 & 1 & 1
\end{array} \\
S_2 &= \begin{array}{c}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & 1 & 1 \\
1 & 1 & 1
\end{array} \\
S_3 &= \begin{array}{c}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & 1 & 1 \\
1 & 1 & 1
\end{array} \\
S_4 &= \begin{array}{c}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & 1 & 1 \\
1 & 1 & 1
\end{array} \\
S_5 &= \begin{array}{c}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & 1 & 1 \\
1 & 1 & 1
\end{array} \\
S_6 &= \begin{array}{c}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & 1 & 1 \\
1 & 1 & 1
\end{array}
\end{align*}$$
and three one-parameter families of two-dimensional simple $\Gamma_0$-representations corresponding to the $Q_0$-representations

$$T_1(\lambda) = \begin{array}{c}
1 \\
0 \\
1
\end{array} \quad T_2(\lambda) = \begin{array}{c}
1 \\
0 \\
1
\end{array} \quad T_3(\lambda) = \begin{array}{c}
1 \\
0 \\
1
\end{array}$$

satisfying $\lambda \neq 1$. With $S_1$ we will denote the set $\{S_1, S_2, S_3, S_4, S_5, S_6\}$ of all one-dimensional $\Gamma_0$-representations.

Consider a finite set $\mathcal{S} = \{V_1, \ldots, V_k\}$ of simple $\Gamma_0$-representations and identify $V_i$ with the corresponding $Q_0$-representation of dimension vector $\alpha_i$. Consider the semi-simple $\Gamma_0$-representation

$$M = V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}$$

The theory of Luna slices allows us to describe the étale local structure of the component $\text{iss}_\beta \Gamma_0$, where $\beta = \sum_i m_i \alpha_i$, in a neighborhood of the point corresponding to $M$. We will assume throughout that $\beta = (a, b; x, y, z)$ is the dimension vector of a $\theta$-stable representation, that is, that $\max(x, y, z) \leq \min(a, b)$.

Let $\mathcal{O}(M)$ be the $GL(\beta) = GL_a \times GL_b \times GL_x \times GL_y \times GL_z$-orbit of $M$ in the representation space $\text{rep}(Q_0, \beta)$, then the normal space to the orbit

$$N_M = \frac{T_M(\text{rep}(Q_0, \beta))}{T_M(\mathcal{O}(M))} \simeq \text{Ext}^1_{Q_0}(M, M)$$

is the extension space, see for example [5] II.2.7. Because

$$\text{Ext}^1_{Q_0}(M, M) = \begin{bmatrix}
M_{m_1} (\text{Ext}^1_{Q_0}(V_1, V_1)) & \ldots & M_{m_1 \times m_k} (\text{Ext}^1_{Q_0}(V_1, V_k)) \\
\vdots & & \vdots \\
M_{m_k \times m_1} (\text{Ext}^1_{Q_0}(V_k, V_1)) & \ldots & M_{m_k} (\text{Ext}^1_{Q_0}(V_k, V_k))
\end{bmatrix}$$

we can identify the vectorspace $\text{Ext}^1_{Q_0}(M, M)$ to the representation space $\text{rep}(Q_S, \alpha_M)$ of the quiver $Q_S$ on $k$ vertices $\{v_1, \ldots, v_k\}$ (vertex $v_i$ corresponding to the simple $\Gamma_0$-representation $V_i$) such that the number of directed arrows from vertex $v_i$ to vertex $v_j$ is equal to

$$\# \{\begin{array}{c} v_i \\
v_j \end{array} \} = \dim_{\mathbb{C}} \text{Ext}^1_{Q_0}(V_i, V_j)$$

and the dimension vector $\alpha_M$ of $Q_S$ is given by the multiplicities of the simple factors in $M$, that is, $\alpha_M = (m_1, \ldots, m_k)$. Observe that the stabilizer subgroup of $M$ is equal to $GL(\alpha_M) = GL_{m_1} \times \cdots \times GL_{m_k}$.

The quiver $Q_S$ is called the local quiver of $M$, see for example [1] or [8], and can be determined from the Euler form of $Q_0$ which is the bilinear map

$$\chi_{Q_0} : \mathbb{Z}^5 \times \mathbb{Z}^5 \rightarrow \mathbb{Z} \quad \chi_{Q_0}(\alpha, \beta) = \alpha.M_{Q_0}.\beta^{tr}$$

determined by the matrix

$$M_{Q_0} = \begin{bmatrix}
1 & 0 & -1 & -1 & -1 \\
0 & 1 & -1 & -1 & -1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}$$
For $V$ and $W$ $Q_0$-representation of dimension vector $\alpha$ and $\beta$, we have
\[ \dim \mathcal{C} \text{Hom}_{Q_0}(V, W) - \dim \mathcal{C} \text{Ext}^1_{Q_0}(V, W) = \chi_{Q_0}(\alpha, \beta) \]
Recall that $\text{Hom}_{Q_0}(V, W) = \delta_{VW} \mathbb{C}$ whenever $V$ and $W$ are $\theta$-stable quiver representations. From the Luna slice theorem we obtain, see for example [8, §4.2].

**Theorem 3.** Let $\mathcal{S} = \{V_1, \ldots, V_k\}$ be a finite set of simple $\Gamma_0$-representations with corresponding $Q_0$-dimension vectors $\alpha_i$. Consider the semi-simple $\Gamma_0$-representation
\[ M = V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k} \]
with $Q_0$-dimension vector $\beta = \sum_i m_i \alpha_i$. Let $Q_{\mathcal{S}}$ be the local quiver described above and let $\alpha_M = (m_1, \ldots, m_k)$ be the $Q_{\mathcal{S}}$-dimension vector determined by the multiplicities. Then, the action map
\[ GL(\beta) \times^{GL(\alpha_M)} \text{rep}(Q_{\mathcal{S}}, \alpha_M) \longrightarrow \text{rep}_\beta \Gamma_0 \]
sending the class of $(g, N)$ in the associated fibre bundle to the representation $g(M + N)$ where $M + N$ is the representation in the normal space to the orbit $\mathcal{O}(M)$ corresponding to the $Q_{\mathcal{S}}$-representation $N$, is a $GL(\beta)$-equivariant étale map with a Zariski dense image. Taking $GL(\beta)$ quotients on both sides, we obtain an étale action map
\[ \text{iss}(Q_S, \alpha_M) \longrightarrow \text{iss}_\beta \Gamma_0 \]
with a Zariski dense image.

2. **The Action Maps for $S_1$ and Rationality**

In order to apply theorem 3 we need to consider a family $\mathcal{S}$ of simple $\Gamma_0$-representations generating a semi-simple representation $M$ in every component $\text{iss}_\beta \Gamma_0$, and, we need to make the action map explicit, that is, we need to identify representations of the quiver $Q_{\mathcal{S}}$ with representations in the normal space to the orbit $\mathcal{O}(M)$.

We consider the set $S_1 = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ of all one-dimensional $\Gamma_0$-representations, using the notations as before. Consider the semi-simple $\Gamma_0$-representation of dimension $n = \sum_i a_i$
\[ M = S_1^{\oplus a_1} \oplus S_2^{\oplus a_2} \oplus S_3^{\oplus a_3} \oplus S_4^{\oplus a_4} \oplus S_5^{\oplus a_5} \oplus S_6^{\oplus a_6} \]
Then $M$ corresponds to a $\theta$-semistable $Q_0$-representation of dimension vector
\[ \beta_M = (a_1 + a_3 + a_5, a_2 + a_4, a_1 + a_4, a_2 + a_5, a_3 + a_6) \]
Under the identification [1] the $n \times n$ matrix $B = (B_{ij})_{1 \leq i \leq 3, 1 \leq j \leq 2}$ determined by $M$ consists of the following block-matrices $B_{ij}$ containing themselves blocks of sizes $a_u \times a_v$ for the appropriate $u$ and $v$
\[
B_{11} = \begin{bmatrix} 1_{a_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1_{a_5} \end{bmatrix} \quad B_{31} = \begin{bmatrix} 0 & 1_{a_3} \\ 0 & 0 \end{bmatrix} \\
B_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1_{a_4} & 0 \end{bmatrix} \quad B_{22} = \begin{bmatrix} 1_{a_2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1_{a_6} \end{bmatrix}
\]
Theorem 4. With $\alpha_M = (a_1, \ldots, a_6)$, the étale action map

$$\text{iss}(Q_{S_i}, \alpha_M) \longrightarrow \text{iss}_{\beta_M} \Gamma_0$$

is induced by sending a representation in $\text{rep}(Q_{S_i}, \alpha_M)$ defined by the matrices

$$B = \begin{bmatrix}
1_{a_1} & 0 & 0 & C_{21} & 0 & C_{61} \\
0 & C_{34} & C_{54} & 0 & 1_{a_4} & 0 \\
C_{12} & C_{32} & 0 & 1_{a_2} & 0 & 0 \\
0 & 0 & 1_{a_5} & 0 & C_{45} & C_{65} \\
0 & 1_{a_3} & 0 & C_{23} & C_{43} & 0 \\
C_{16} & 0 & C_{56} & 0 & 0 & 1_{a_6}
\end{bmatrix}$$

Under this map, simple $Q_{S_i}$-representations with invertible matrix $B$ are mapped to irreducible $n$-dimensional $\Gamma_0$-representations.

Hence, if the coefficients in the matrices $C_{ij}$ give a parametrization of (an open set of) the quotient variety $\text{iss}(Q_{S_i}, \alpha_M)$, the $n$-dimensional representations of the three string braid group $B_3$ given by

$$\begin{cases}
\sigma_1 \mapsto \lambda B^{-1} \begin{bmatrix}
1_{a_1+a_4} & 0 & 0 \\
0 & \rho^2 1_{a_2+a_5} & 0 \\
0 & 0 & \rho 1_{a_3+a_6}
\end{bmatrix} B \begin{bmatrix}
1_{a_1+a_4+a_5} & 0 \\
0 & -1_{a_2+a_4+a_6}
\end{bmatrix} \\
\sigma_2 \mapsto \lambda \begin{bmatrix}
1_{a_1+a_4+a_5} & 0 \\
0 & -1_{a_2+a_4+a_6}
\end{bmatrix} B^{-1} \begin{bmatrix}
1_{a_1+a_4} & 0 & 0 \\
0 & \rho^2 1_{a_2+a_5} & 0 \\
0 & 0 & \rho 1_{a_3+a_6}
\end{bmatrix} B
\end{cases}$$

contain a Zariski dense set of the simple $B_3$-representations in $\text{iss}_{\beta_M} B_3$.

Proof. Using the Euler-form of the quiver $Q_0$ and the $Q_0$-dimension vectors of the representations $S_i$ one verifies that the quiver $Q_{S_i}$ is the one given in the statement of the theorem. To compute the components of the tangent space in $M$ to the orbit, take $\text{Lie}(GL(\beta_M))$ as the set of matrices (in block-matrices of sizes $a_u \times a_v$)
and hence the tangent space to the orbit is computed using the action of $GL(\beta_M)$ on the quiver-representations, giving for example for the $B_{11}$-arrow

$$\left( \begin{array}{cc} 1_{a_1} & 0 \\ 0 & 1_{a_3} \end{array} \right) + \epsilon \left( \begin{array}{cc} A'_1 & A'_{14} \\ A_{41} & A'_4 \end{array} \right) \cdot \left( \begin{array}{cc} 1_{a_1} & 0 \\ 0 & 1_{a_3} \end{array} \right) \cdot \left( \begin{array}{cc} 1_{a_1} & 0 \\ 0 & 1_{a_3} \end{array} \right) \cdot \left( \begin{array}{cccc} 1_{a_1} & 0 & 0 & 0 \\ 0 & 1_{a_3} & 0 & 0 \\ 0 & 0 & 1_{a_5} & -\epsilon \end{array} \right) \cdot \left( \begin{array}{cccc} A_1 & A_{13} & A_{15} \\ A_{31} & A_3 & A_{35} \\ A_{51} & A_{53} & A_5 \end{array} \right)$$

which is equal to

$$\left[ \begin{array}{cc} 1_{a_1} & 0 \\ 0 & 0 \end{array} \right] + \epsilon \left[ \begin{array}{cc} A_1 - A'_1 & -A_{13} & -A_{15} \\ 0 & A_{41} & 0 \end{array} \right]$$

and, similarly, the $\epsilon$-components of $B_{21}, B_{31}, B_{12}, B_{22}$ resp. $B_{23}$ are calculated to be

$$B_{21} : \left[ \begin{array}{cccc} 0 & 0 & A_{25} \\ -A_{51} & -A_{53} & A_5 - A'_5 \end{array} \right] \quad B_{31} : \left[ \begin{array}{cccc} -A_{31} & A_3 - A'_3 & -A_{35} \\ 0 & 0 & A_{63} \end{array} \right]$$

$$B_{12} : \left[ \begin{array}{ccc} 0 & A_{41} & 0 \\ -A_{42} & A_4 - A'_4 & -A_{46} \end{array} \right] \quad B_{22} : \left[ \begin{array}{ccc} A_2 - A'_2 & -A_{24} & -A_{26} \\ A_{52} & 0 & 0 \end{array} \right]$$

$$B_{23} : \left[ \begin{array}{ccc} 0 & 0 & A_{36} \\ -A_{62} & -A_{64} & A_6 - A'_{6} \end{array} \right]$$

Here the zero blocks correspond precisely to the matrices $C_{ij}$ describing a representation in $\text{rep}(Q_{S_i}, \alpha_M)$ which can therefore be identified with the normal space in $M$ to the orbit $O(M)$. Here we use the inproduct on $T_M \text{rep}_{\beta_M} \Gamma_0 = \text{rep}_{\beta_M} \Gamma_0$ defined for all $B = (B_{ij})$ and $B' = (B'_{ij})$

$$\langle B, B' \rangle = Tr(B^{tr}B')$$

Hence, the representation $M + N$ is determined by the matrices

$$B_{11} = \left[ \begin{array}{ccc} 1_{a_1} & 0 & 0 \\ 0 & C_{34} & C_{34} \end{array} \right] \quad B_{21} = \left[ \begin{array}{ccc} C_{12} & C_{32} & 0 \\ 0 & 0 & 1_{a_3} \end{array} \right]$$

$$B_{31} = \left[ \begin{array}{ccc} 0 & 1_{a_3} & 0 \\ C_{16} & 0 & C_{56} \end{array} \right] \quad B_{12} = \left[ \begin{array}{ccc} C_{21} & 0 & C_{61} \\ 0 & 1_{a_4} & 0 \end{array} \right]$$

$$B_{22} = \left[ \begin{array}{ccc} 1_{a_2} & 0 & 0 \\ 0 & C_{45} & C_{65} \end{array} \right] \quad B_{32} = \left[ \begin{array}{ccc} C_{23} & C_{43} & 0 \\ 0 & 0 & 1_{a_6} \end{array} \right]$$

To any $Q_0$-representation $V_B$ of dimension vector $\alpha = (a; b; x, y, z)$, with invertible $n \times n$ matrix $B$ corresponds, via the identifications given in the previous section, the $n$-dimensional representation of the modular group $\Gamma_0$ defined by

$$S \mapsto \left[ \begin{array}{c} 1_a \\ 0 \end{array} \right] \quad 0 \quad -1_b$$

$$T \mapsto B^{-1} \left[ \begin{array}{ccc} 1_x & 0 & 0 \\ 0 & \rho y & 0 \\ 0 & 0 & \rho^2 z \end{array} \right] \quad B$$

As $S = \sigma_1 \sigma_2 \sigma_1$ and $T = \sigma_1 \sigma_2$ it follows that $\sigma_1 = T^{-1} S$ and $\sigma_2 = ST^{-1}$. Therefore, when $V_B$ is $\theta$-stable it determines a $C^*$-family of $n$-dimensional representations of
the three string braid group $B_3$ given by

$$
\begin{align*}
\sigma_1 & \mapsto \lambda B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \begin{bmatrix} 1_a & 0 & 0 \\ 0 & -1_b & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} \\
\sigma_2 & \mapsto \lambda \begin{bmatrix} 1_a & 0 & 0 \\ 0 & 0 & \rho 1_z \\ 0 & -1_b & 0 \end{bmatrix} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B
\end{align*}
$$

Using the dimension vector $\beta_M$ and the matrices $B_{ij}$ found before, we obtain the required $B_3$-representations. The final statements follow from theorem 3.

The method of proof indicates how one can make the action maps explicit for any given finite family of irreducible $\Gamma_0$-representations. Note that the condition for $\beta_M$ to be the dimension vector of a $\theta$-stable representation is equivalent to the condition on $\alpha_M$

$$
a_i \leq a_{i-1} + a_{i+1} \quad \text{for all } i \mod 6
$$

which is the condition for $\alpha_M = (a_1, \ldots, a_6)$ to be the dimension vector of a simple $Q_{S_1}$-representation, by [7].

**Theorem 5.** For any $Q_0$-dimension vector $\beta = (a, b; x, y, z)$ admitting a $\theta$-stable representation, that is satisfying $\max(x, y, z) \leq \min(a, b)$ there exist semi-simple representations $M \in \text{iss}_\beta \Gamma_0$ such that for $\alpha_M = (a_1, \ldots, a_6)$ the quotient variety $\text{iss}(Q_{S_1}, \alpha_M)$ is rational. As a consequence, the étale action map

$$
\text{iss}(Q_{S_1}, \alpha_M) \longrightarrow \text{iss}_\beta \Gamma_0
$$

given by theorem 4 determines a rational parametrization of a Zariski dense subset of $\text{iss}_\beta B_3$.

**Proof.** The condition on $\beta$ ensures that there are simple $\alpha_M$-dimensional representations of $Q_{S_1}$, and hence that $\alpha_M$ is a Schur root. By a result of Aidan Schofield [13] this implies that the quotient variety $\text{iss}(Q_{S_1}, \alpha_M)$ is birational to the quotient variety of $p$-tuples of $h \times h$ matrices under simultaneous conjugation, where

$$
h = \gcd(a_1, \ldots, a_6) \quad \text{and} \quad p = 1 - \chi_{Q_{S_1}}(\frac{\alpha_M}{h}, \frac{\alpha_M}{h}) = 1 + \frac{1}{h^2} \left(2 \sum_{i=1}^{6} a_i a_{i+1} - \sum_{i=1}^{6} a_i^2 \right)
$$

By Procesi’s result [11, Prop. IV.6.4] and the known rationality results (see for example [2]) the result follows when we can find such an $\alpha_M$ satisfying $\gcd(a_1, \ldots, a_6) \leq 4$. For small dimensions one verifies this by hand, and, for larger dimensions having found a representation $M$ with $\gcd(a_1, \ldots, a_6) > 4$, one can
modify the multiplicities of the simple components by

\[
\begin{array}{cccc}
2 & \text{or} & 3 \\
1 & 1 & 1 & 2 \\
-1 & -1 & -2 & -1 \\
-2 & -3 & & \\
\end{array}
\]

(or a cyclic permutation of these) to obtain another semi-simple representation \(M'\) lying in the same component, but having \(gcd(\alpha_{M'}) = 1.\)

\[\square\]

3. Rational dense families for \(n < 12\)

In this section we will determine for every irreducible component \(iss_\beta \Gamma_0,\) with \(\beta = (a, b; x, y, z)\) such that \(a + b = x + y + z = n < 12\) satisfying \(max(x, y, z) \leq min(a, b),\) a semi-simple \(\Gamma_0\)-representation

\[M_\beta = S_1^{a_1} \oplus S_2^{a_2} \oplus S_3^{a_3} \oplus S_4^{a_4} \oplus S_5^{a_5} \oplus S_6^{a_6}\]

contained in \(iss_\beta \Gamma_0.\) Then, we will explicitly determine a rational family of representations in \(\text{rep}(Q_{S_1}, \alpha_M)\) for \(\alpha_M = (a_1, \ldots, a_6).\)

As we are only interested in the corresponding \(B_3\)-representations we may assume that \(\beta = (a, b; x, y, z)\) is such that \(a \geq b\) and \(x \geq y \geq z.\) Observe that in going from \(\Gamma_0\)-representations to \(B_3\)-representations we multiply with \(\lambda \in \mathbb{C}^*.\) Hence, there is a \(\mu_6\)-action on the dimension vectors of \(Q_0\)-representations giving the same component of \(B_3\)-representations. That is, we have to consider only one dimension vector from the \(\mu_6\)-orbit

\[
(a, b; x, y, z) \to (b, a; z, x, y) \to (a, b; y, z, x)
\]

\[
(b, a; y, z, x) \leftarrow (a, b; z, x, y) \leftarrow (b, a; x, y, z)
\]

Hence we may assume that \(a \geq b\) and that \(x = max(x, y, z).\) If we find a module \(M_\beta\) and corresponding rational parametrization for \(\beta = (a, b; x, y, z)\) by representations in \(\text{rep}(Q_{S_1}, \alpha_M),\) then we can mirror that description to obtain similar data for
\( \beta' = (a, b; x, z, y) \) via

\[
\begin{array}{cccccc}
C_1 & C_6 & C_{12} & C_4 & C_{16} & C_{21} \\
C_{61} & C_{16} & C_2 & C_{6} & C_{12} & C_{21} \\
C_{23} & C_{32} & C_3 & C_{32} & C_2 & C_3 \\
C_{43} & C_{4} & C_{43} & C_{4} & C_4 & C_4 \\
C_{54} & C_{56} & C_{56} & C_{56} & C_{56} & C_{54} \\
C_{65} & C_{55} & C_{55} & C_{55} & C_{55} & C_{55} \\
& & & & & \\
C_1 & C_6 & C_{12} & C_4 & C_{16} & C_{21} \\
C_{61} & C_{16} & C_2 & C_{6} & C_{12} & C_{21} \\
C_{23} & C_{32} & C_3 & C_{32} & C_2 & C_3 \\
C_{43} & C_{4} & C_{43} & C_{4} & C_4 & C_4 \\
C_{54} & C_{56} & C_{56} & C_{56} & C_{56} & C_{54} \\
C_{65} & C_{55} & C_{55} & C_{55} & C_{55} & C_{55} \\
& & & & & \\
\end{array}
\]

proving that we may indeed restrict to \( \beta = (a, b; x, y, z) \) with \( a \geq b \) and \( x \geq y \geq z \).

**Lemma 1.** The following representations provide a rational family determining a dense subset of the quotient varieties \( \text{iss}(Q_{S_1}, \alpha) \) for these (minimalistic) dimension vectors \( \alpha \).

- **(A)**: \[
\begin{bmatrix}
1 \\
a
\end{bmatrix}
\]
- **(B)**: \[
\begin{bmatrix}
0 & b \\
1 & c
\end{bmatrix}
\]
- **(C)**: \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]
- **(D)**: \[
\begin{bmatrix}
a & 0 \\
0 & b
\end{bmatrix}
\]
- **(E)**: \[
\begin{bmatrix}
a & 1 & b \\
1 & 0 & 0 \\
0 & 1 & 1
\end{bmatrix}
\]

**Proof.** Recall from [7] that the rings of polynomial quiver invariants are generated by traces along oriented cycles in the quiver. It also follows from [7] that these dimension vectors are the dimension vectors of simple representations and that their quotient varieties are rational of transcendence degrees: 1(A), 3(B), 4(C), 5(D) and 5(E). This proves \( (A) \). Cases \( (B), (C) \) and \( (D) \) are easily seen (focus on the middle vertex) to be equivalent to the problem of classifying couples of \( 2 \times 2 \) matrices \( (A, B) \) up to simultaneous conjugation in case \( (D) \). For \( (C) \) the matrix \( A \) needs to have rank one and for \( (B) \) both \( A \) and \( B \) have rank one. It is classical that the corresponding rings of polynomial invariants are: \( (B) \mathbb{C}[\text{Tr}(A), \text{Tr}(B), \text{Tr}(AB)] \), \( (C) \mathbb{C}[\text{Tr}(A), \text{Tr}(B), \text{Det}(B), \text{Tr}(AB)] \) and
In case (E), as $3 \geq 1, 2$ we can invoke the first fundamental theorem for $GL_n$-invariants (see [5, Thm. II.4.1]) to eliminate the $GL_3$-action by composing arrows through the 3-vertex. This reduces the study to the quiver-representations below, of which the indicated representations form a rational family by case (D)

\[
\begin{bmatrix}
1 \\
0 \\
1 & b
\end{bmatrix}
\quad \quad
\begin{bmatrix}
c & d \\
e & 1
\end{bmatrix}
\]

and one verifies that the indicated family of (E) representations does lead to these matrices.

**Lemma 2.** Adding one vertex at a time (until one has at most 5 vertices) to either end of the full subquivers of $Q_{S_i}$ of lemma [7] one can extend the dimension vector and the rational dense family of representations as follows in these allowed cases

- $\begin{bmatrix} 1 \\ u \end{bmatrix}$
- $\begin{bmatrix} v & w \end{bmatrix}$
- $\begin{bmatrix} a & v & w \end{bmatrix}$

**Proof.** Before we add an extra vertex, the family of representations is simple and hence the stabilizer subgroup at the end-vertex is reduced to $\mathbb{C}^*$. Adding the vertex and arrows, we can quotient out the base-change action at the new end-vertex by composing the two new arrows (the 'first fundamental theorem for $GL_n$-invariant theory, see [5, Thm. II.4.1]). As a consequence, we have to classifying resp. a $1 \times 1$ matrix, a rank one $2 \times 2$ matrix or a $2 \times 2$ matrix with trivial action. The given representations do this.

Assuming the total number of vertices is $\leq 5$ we can 'glue' two such subquivers and families of representations at a common end-vertex having dimension 1. Indeed, the ring of polynomial invariants of the glued quivers is the tensor product of those of the two subquivers as any oriented cycle passing through the glue-vertex can be decomposed into the product of two oriented cycles, one belonging to each component. When the two subquivers $Q$ and $Q'$ are glued along a 1-vertex located at vertex $i$ we will denote the new subquiver $Q \bullet_i Q'$. The problem of 'closing-the-circle' is solved by adding the 6-th vertex to one of these families of representations on the remaining five vertices.
Lemma 3. Assume we have one of these dimension vectors on the full subquiver of \( Q_{S_1} \) on five vertices and the corresponding family of generically simple representations. Then, we can extend the dimension vector and the rational family of representations to the full quiver \( Q_{S_1} \).

\[
\begin{array}{ccc}
1 & v & w \\
\downarrow & & \downarrow \\
u & \cdot & \cdot \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & \cdot & \cdot \\
\downarrow & & \downarrow \\
u & \cdot & \cdot \\
\end{array}
\]

\[
\begin{array}{ccc}
v & w & \cdot \\
\downarrow & & \downarrow \\
p & q & \cdot \\
\end{array}
\]

Proof. Forgetting the right-hand arrows, the left-hand representations are added as in the previous lemma. Then, the action on the right-hand arrows is trivial. \(\square\)

For any of the obtained \( Q_{S_1} \)-dimension vectors, we can now describe a rational family of generically simple representations via a code containing the following ingredients:

- \( A, B, C, D, E \) will be representations as in lemma 1 and \( E \) will denote the mirror images of \( C \) and \( E \)
- we add 1’s or 2’s when we add these dimensions to the appropriate side as in lemma 2
- we add \( i \) when we glue along a 1-vertex placed at spot \( i \), and
- we add \( 1_j \) if we close-up the family at a 1-vertex placed at spot \( j \) as in lemma 3

Theorem 6. In Figure 1 we list rational families for all irreducible components of \( \text{iss}_n B_3 \) for dimensions \( n < 12 \). An + -sign in column two indicates there are two such components, mirroring each other, \( \beta_M \) indicates the \( Q_0 \)-dimension vector and \( \alpha_M \) gives the multiplicities of the \( S_1 \)-simples of a semi-simple \( \Gamma_0 \)-representation in the component. The last column gives the dimension of this \( \text{iss}_n B_3 \)-component.

Proof. We are looking for \( Q_0 \)-dimension vectors \( \beta = (a, b; x, y, z) \) satisfying \( a + b = x + y + z = n \), \( \max(x, y, z) \leq \min(a, b) \) and \( a \geq b, x \geq y \geq z \) (if \( y \neq z \) there is a mirror component). One verifies that one only obtains the given 5-tuples and that the indicated \( Q_{S_1} \)-dimension vectors \( \alpha_M \) are compatible with \( \beta \) and that the code is allowed by the previous lemmas. Only in the final entry, type 11e we are forced to close-up in a 2-vertex, but by the argument given in lemma 3 it follows that in
In this case the following representations

\[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
u & v \\
w & x
\end{pmatrix}
\begin{pmatrix}
y \\
z
\end{pmatrix}
\begin{pmatrix}
p \\
q
\end{pmatrix}
\]

will extend the family to a rational dense family in \(\text{iss}(Q_{S_1}, \alpha_M)\).
For example, the code $13\overline{c} \cdot \overline{c} C$ of component $10c$ will denote the following rational family of $Q_{S_3}$-representations

$$B_3 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & b & c & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & b & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & l & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & m & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & k & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

which give us a rational $16 = 15 + 1$-dimensional family of $B_3$-representations, Zariski dense in their component.

4. Detecting knot vertibility

The rational Zariski dense families of $B_3$-representations can be used to separate conjugacy classes of three string braids by taking traces. Recall from [3] that a flype is a braid of the form

$$b = \sigma_1^u \sigma_2^v \sigma_1^w \sigma_2^\epsilon$$

where $u, v, w \in \mathbb{Z}$ and $\epsilon = \pm 1$. A flype is said to be non-degenerate if $b$ and the reversed braid $b' = \sigma_2^v \sigma_1^u \sigma_2^v \sigma_1^u$ are in distinct conjugacy classes. An example of a
non-degenerate flype of minimal length is $b = \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$

Hence we can ask whether a 6-dimensional $B_3$-representation can detect that $b$ lies in another conjugacy class than its reversed braid $b' = \sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}$

Note that from the classification of all simple $B_3$-representation of dimension $\leq 5$ by Tuba and Wenzl [14] no such representation can separate $b$ from $b'$. Let us consider 6-dimensional $B_3$-representations. From the previous section we retain that there are 4 irreducible components in $\text{iss}_6 B_3$ (two being mirror images of each other) and that rational parametrizations for the corresponding $\Gamma_0$-components are given by the following representations of type 6a, 6b and 6c

As a consequence we obtain a rational Zariski dense subset of $\text{iss}_6 B_3$ from theorem 4 using the following matrices $B$ for the three components

Using a computer algebra system, for example SAGE [12], one verifies that $\text{Tr}(b) = \text{Tr}(b')$ for all 6-dimensional $B_3$-representations belonging to components 6a and 6c. However, $\text{Tr}(b) \neq \text{Tr}(b')$ for an open subset of representations in component 6b and
hence 6-dimensional $B_3$-representations can detect non-degeneracy of flypes. The specific representation given in the introduction (obtained by specializing $a = c = e = g = 1$ and $b = d = f = \lambda = -1$ gives $\text{Tr}(b) = 648\rho - 228$ whereas $\text{Tr}(b') = -648\rho - 876$. In order to use the family of 6-dimensional $B_3$-representations as an efficient test to separate 3-braids from their reversed braids it is best to specialize the variables to random integers in $\mathbb{Z}[\rho]$ to obtain an 8-parameter family of $B_3$-representations over $\mathbb{Q}(\rho)$.

One can then test the ability of this family to separate braids from their reversed braid on the list of all knots having at most 8 crossings and being closures of three string braids, as provided by the Knot Atlas [4]. It turns out that the braids of the following knots can be separated from their reversed braids: $6_3$, $7_5$, $8_7$, $8_9$, $8_{10}$ (all flypes), as well as the smallest non-invertible knot $8_{17}$ as mentioned in the introduction.

**Remark 1.** The referee suggested an alternative point of view. Taking transposes of the representing matrices

$$(\sigma_1, \sigma_2) \longrightarrow (\sigma_1^{tr}, \sigma_2^{tr})$$

gives an involution on $\text{rep}_{B_3}$ which passes to an involution on $\text{iss}_{n,B_3}$. The results of this paper show that this involution is trivial whenever $n \leq 5$ and for $n = 6$ it is non-trivial for the 8-dimensional component. It would be interesting to determine the dimensions of the fixed point varieties, for general $n$.

**References**


