THE GEOMETRY OF REPRESENTATIONS OF
3-DIMENSIONAL SKLYANIN ALGEBRAS

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Abstract. The representation scheme $\text{rep}_A$ of the 3-dimensional Sklyanin algebra $A$ associated to a plane elliptic curve and $n$-torsion point contains singularities over the augmentation ideal $m$. We investigate the semi-stable representations of the noncommutative blow-up algebra $B = A \oplus m t \oplus m^2 t^2 \oplus \ldots$ to obtain a partial resolution of the central singularity

$$\text{proj } \mathbb{Z}(B) \rightarrow \text{spec } \mathbb{Z}(A)$$

such that the remaining singularities in the exceptional fiber determine an elliptic curve and are all of type $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$.

1. Introduction

Three dimensional Sklyanin algebras appear in the classification by M. Artin and W. Schelter [2] of graded algebras of global dimension 3. In the early 90ties this class of algebras was studied extensively by means of noncommutative projective algebraic geometry, see a.o. [3], [4], [5], [9] and [15]. Renewed interest in this class of algebras arose recently as they are superpotential algebras and as such relevant in supersymmetric quantum field theories, see a.o. [6] and [16].

Consider a smooth elliptic curve $E$ in Hesse normal form $V((a^3 + b^3 + c^3)XYZ - abc(X^3 + Y^3 + Z^3)) \rightarrow \mathbb{P}^2$ and the point $p = [a : b : c]$ on $E$. The 3-dimensional Sklyanin algebra $A$ corresponding to the pair $(E, p)$ is the noncommutative algebra with defining equations

$$\begin{align*}
ax y + by x + cz^2 &= 0 \\
ay z + bx y + cx^2 &= 0 \\
ax x + bx z + cy^2 &= 0
\end{align*}$$

The connection comes from the fact that the multi-linearization of these equations defines a closed subscheme in $\mathbb{P}^2 \times \mathbb{P}^2$ which is the graph of translation by $p$ on the elliptic curve $E$, see [4]. Alternatively, one obtains the defining equations of $A$ from the superpotential $W = axyz + byzx + \frac{c}{3}(x^3 + y^3 + z^3)$, see [16].

The algebra $A$ has a central element of degree 3, found by computer search in [2]

$$c_3 = c(a^3 - c^3)x^3 + a(b^3 - c^3)xyz + b(c^3 - a^3)yxz + c(e^3 - b^3)y^3$$

with the property that $A/(c_3)$ is the twisted coordinate ring of the elliptic curve $E$ with respect to the automorphism given by translation by $p$, see [4]. We will prove an intrinsic description of this central element, answering a MathOverflow question [8].
**Theorem 1.** The central element $c_3$ of the 3-dimensional Sklyanin algebra $A$ corresponding to the pair $(E, p)$ can be written as
\[ c(a^3 - b^3)(xyz + yzx + zxy) + b(c^3 - a^3)(yuz + xzy + zyx) + c(a^3 - b^3)(x^3 + y^3 + z^3) \]
and is the superpotential of the 3-dimensional Sklyanin algebra $A'$ corresponding to the pair $(E, [-2]p)$.

Next, we turn to the study of finite dimensional representations of $A$ which is important in supersymmetric gauge theory as they correspond to the vacua states. It is well known that $A$ is a finite module over its center $Z(A)$ and a maximal order in a central simple algebra of dimension $n^2$ if and only if the point $p$ is of finite order $n$, see [4]. We will further assume that $(n,3) = 1$ in which case J. Tate and P. Smith proved in [15] that the center $Z(A)$ is generated by $c_3$ and the reduced norms of $x, y$ and $z$ (which are three degree $n$ elements, say $x', y', z'$) satisfying one relation of the form
\[ c_3^3 = \text{cubic}(x', y', z') \]
It is also known that $\text{proj} Z(A) \simeq \mathbb{P}^2$ with coordinates $[x' : y' : z']$ in which the cubic$(x', y', z')$ defines the isogenous elliptic curve $E' = E/\langle p \rangle$, see a.o. [9]. We will use these facts to give explicit matrices for the simple $n$-dimensional representations of $A$ and show that $A$ is an Azumaya algebra away from the isolated central singularity.

However, the scheme $\text{rep}_n A$ of all (trace preserving) $n$-dimensional representations of $A$ contains singularities in the nullcone. We then try to resolve these representation singularities by considering the noncommutative analogue of a blow-up algebra
\[ B = A \oplus \mathfrak{m}t \oplus \mathfrak{m}^2t^2 \oplus \ldots \subset A[t, t^{-1}] \]
where $\mathfrak{m} = (x, y, z)$ is the augmentation ideal of $A$. We will prove

**Theorem 2.** The scheme $\text{rep}_n^\text{ss} B$ of all semi-stable $n$-dimensional representations of the blow-up algebra $B$ is a smooth variety.

This allows us to compute all the (graded) local quivers in the closed orbits of $\text{rep}_n^\text{ss} B$ as in [10] and [7]. This information then leads to the main result of this paper which gives a partial resolution of the central isolated singularity.

**Theorem 3.** The exceptional fiber $\mathbb{P}^2$ of the canonical map
\[ \text{proj} Z(B) \longrightarrow \text{spec} Z(A) \]
contains $E' = E/\langle p \rangle$ as the singular locus of $\text{proj} Z(B)$. Moreover, all these singularities are of type $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$ with $\mathbb{C}^2/\mathbb{Z}_n$ an Abelian quotient surface singularity.

2. **Central elements and superpotentials**

The finite Heisenberg group of order 27
\[ \langle u, v, w \mid [u, v] = w, \ [u, w] = [v, w] = 1, \ w^3 = v^3 = u^3 = 1 \rangle \]
has a 3-dimensional irreducible representation $V = \mathbb{C}x + \mathbb{C}y + \mathbb{C}z$ given by the action
\[
\begin{align*}
  u &\mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
  v &\mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{bmatrix} \\
  w &\mapsto \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{bmatrix}
\end{align*}
\]
One verifies that $V \otimes V$ decomposes as three copies of $V^*$, that is,

$$V \otimes V \simeq \wedge^2(V) \oplus S^2(V) \simeq V^* \oplus (V^* \oplus V^*)$$

where the three copies can be taken to be the subspaces

\[
\begin{align*}
V_1 &= \mathbb{C}(yz - zy) + \mathbb{C}(zx - xz) + \mathbb{C}(xy - yx) \\
V_2 &= \mathbb{C}(yz + zy) + \mathbb{C}(zx + xz) + \mathbb{C}(xy + yx) \\
V_3 &= \mathbb{C}x^2 + \mathbb{C}y^2 + \mathbb{C}z^2
\end{align*}
\]

Taking the quotient of $\mathbb{C}(x, y, z)$ modulo the ideal generated by $V_1 = \wedge^2 V$ gives the commutative polynomial ring $\mathbb{C}[x, y, z]$. Hence we can find analogues of the polynomial ring in three variables by dividing $\mathbb{C}(x, y, z)$ modulo the ideal generated by another copy of $V^*$ in $V \otimes V$ and the resulting algebra will inherit an action by $H_3$. Such a copy of $V^*$ exists for all $[A : B : C] \in \mathbb{P}^2$ and is spanned by the three vectors

\[
\begin{align*}
A(yz - zy) + B(yz + zy) + Cx^2 \\
A(zx - xz) + B(zx + xz) + Cy^2 \\
A(xy - yx) + B(xy + yx) + Cz^2
\end{align*}
\]

and by taking $a = A + B, b = B - A$ and $c = C$ we obtain the defining relations of the 3-dimensional Sklyanin algebra. In particular there is an $H_3$-action on $A$ and the canonical central element $c_3$ of degree 3 must be a 1-dimensional representation of $H_3$. It is obvious that $c_3$ is fixed by the action of $v$ and a minor calculation shows that $c_3$ is also fixed by $u$. Therefore, the central element $c_3$ given above, or rather $3c_3$, can also be represented as

$$a(b^3 - c^3)(xyz + yzx + zxy) + b(c^3 - a^3)(ytx + xzy + zy) + c(a^3 - b^3)(x^3 + y^3 + z^3)$$

Now, let us reconsider the superpotential $W = axy + byx + \frac{c}{3}(x^3 + y^3 + z^3)$ for a $[a : b : c] \in \mathbb{P}^2$. This superpotential gives us three quadratic relations by taking cyclic derivatives with respect to the variables

\[
\begin{align*}
\partial_x W &= ayz + bzy + cx^2 \\
\partial_y W &= axz + bxz + cy^2 \\
\partial_z W &= axy + byx + cz^2
\end{align*}
\]

giving us the defining relations of the 3-dimensional Sklyanin algebra. We obtain the same equations by considering a more symmetric form of $W$, or rather of $3W$

$$a(xyz + yzx + zxy) + b(yxz + xzy + zyx) + c(x^3 + y^3 + z^3)$$

We see that the form of the central degree 3 element and of the superpotential are similar but with different coefficients. This means that the central element is the superpotential defining another 3-dimensional Sklyanin algebra and theorem 1 clarifies this connection.

**Proof of Theorem 1**: The 3-dimensional Sklyanin algebra determined by the superpotential $3c_3$ is determined by $[a(b^3 - c^3) : b(c^3 - a^3) : c(a^3 - b^3)]$ (instead of $[a : b : c]$ for the original). Therefore, the associated elliptic curve has defining Hesse equation

$$V(\alpha(x^3 + y^3 + z^3) - \beta xyz) \mathbb{P}^2$$
where
\[
\begin{align*}
\alpha &= a(b^3 - c^3)b(c^3 - a^3)c(a^3 - b^3) \\
\beta &= (a(b^3 - c^3))^3 + (b(c^3 - a^3))^3 + (c(a^3 - b^3))^3
\end{align*}
\]
but this is the same equation, up to a scalar, as the original curve
\[
E = \mathcal{V}(abc(x^3 + y^3 + z^3) - (a^3 + b^3 + c^3)xyz)
\]
The tangent line to $E$ in the point $p = [a : b : c]$ has equation
\[
\mathcal{V}((2a^3bc - b^4c - bc^4)(x - a) + (2ab^3c - a^4c - ac^4)(y - b) + (2abc - a^4b - ab^4)(z - c))
\]
and so the third point of intersection is
\[
[-2]p = [a(b^3 - c^3) : b(c^3 - a^3) : c(a^3 - b^3)]
\]
which are the parameters of the algebra. $\square$

3. Resolving representation singularities

Let $R$ be a graded $\mathbb{C}$-algebra, generated by finitely many elements $x_1, \ldots, x_m$ where $\text{deg}(x_i) = d_i \geq 0$, which is a finite module over its center $Z(R)$. Following [14] we say that $R$ is a Cayley-Hamilton algebra of degree $n$ if there is a $Z(R)$-linear gradation preserving trace map $\text{tr} : R \rightarrow Z(R)$ such that for all $a, b \in R$ we have
\[
\begin{align*}
\text{tr}(ab) &= \text{tr}(ba) \\
\text{tr}(1) &= n \\
\chi_{n,a}(a) &= 0
\end{align*}
\]
where $\chi_{n,a}(t)$ is the $n$-th Cayley-Hamilton identity expressed in the traces of powers of $a$. Maximal orders in a central simple algebra of dimension $n^2$ are examples of Cayley-Hamilton algebras of degree $n$.

In particular, a 3-dimensional Sklyanin algebra $A$ associated to a couple $(E, p)$ where $p$ is a torsion point of order $n$, and the corresponding blow-up algebra $B = A \oplus \mathfrak{m}^t \oplus \mathfrak{m}^t \circ \mathfrak{m}^t \oplus \ldots$ are affine graded Cayley-Hamilton algebras of degree $n$ equipped with the (gradation preserving) reduced trace map.

If $R$ is an affine graded Cayley-Hamilton algebra of degree $n$ we define $\text{rep}_n R$ to be the affine scheme of all $n$-dimensional trace preserving representations, that is all algebra morphisms
\[
R \rightarrow M_n(\mathbb{C}) \quad \text{such that} \quad \forall a \in R : \phi(\text{tr}(a)) = \text{Tr}(\phi(a))
\]
where $\text{Tr}$ is the usual trace map on $M_n(\mathbb{C})$. Isomorphism of representations defines a $\text{GL}_n$-action of $\text{rep}_n R$ and a result of Artin’s [1] asserts that the closed orbits under this action, that is the points of the GIT-quotient scheme $\text{rep}_n R/\text{GL}_n$, are precisely the isomorphism classes of $n$-dimensional trace preserving semi-simple representations of $R$. The reconstruction result of Procesi [14] asserts that in this setting
\[
\text{spec } Z(R) \simeq \text{rep}_n R/\text{GL}_n
\]
The gradation on $R$ defines an additional $\mathbb{C}^*$-action on $\text{rep}_n R$ commuting with the $\text{GL}_n$-action. With $\text{rep}_n^{ss} R$ we denote the Zariski open subset of all semi-stable trace preserving representations $\phi : R \rightarrow M_n(\mathbb{C})$, that is, such that there is an homogeneous central element $c$ of positive degree such that $c(\phi) \neq 0$. We have the following graded version of Procesi’s reconstruction result, see a.o. [7]
\[
\text{proj } Z(R) \simeq \text{rep}_n^{ss} R/\text{GL}_n \times \mathbb{C}^*$
As a $\mathcal{G}L_n \times \mathbb{C}^*$-orbit is closed in $\text{rep}^{{\text{ss}}} R$ if and only if the $\mathcal{G}L_n$-orbit is closed we see that points of $\text{proj} Z(R)$ classify one-parameter families of isoclasses of trace-preserving $n$-dimensional semi-simple representations of $R$. In case of a simple representation such a one-parameter family determines a graded algebra morphism

$$R \longrightarrow M_n(\mathbb{C}[t, t^{-1}])((0, \ldots, 0, 1, \ldots, 1, e^{-1}, \ldots, e^{-1}))$$

where $e$ is the degree of $t$ and where we follow [13] in defining the shifted graded matrix algebra $M_n(S)(a_1, \ldots, a_n)$ by taking is homogeneous part of degree $i$ to be

$$\begin{bmatrix}
S_i & S_{i-a_1+a_2} & \cdots & S_{i-a_1+a_n} \\
S_{i-a_2+a_1} & S_i & \cdots & S_{i-a_2+a_n} \\
\vdots & \vdots & \ddots & \vdots \\
S_{i-a_n+a_1} & S_{i-a_n+a_2} & \cdots & S_i
\end{bmatrix}$$

The $\mathcal{G}L_n \times \mathbb{C}^*$-stabilizer subgroup of any of the simples $\phi$ in this family is then isomorphic to $\mathbb{C}^* \times \mu_e$ where the cyclic group $\mu_e$ has generator $(g_\zeta, \zeta) \in \mathcal{G}L_n \times \mathbb{C}^*$ where $\zeta$ is a primitive $e$-th root of unity and

$$g_\zeta = \text{diag}(1, \ldots, 1, \zeta, \ldots, \zeta, \ldots, \zeta^{e-1}, \ldots, \zeta^{e-1})$$

see [7, lemma 4]. If, in addition, $\phi$ is a smooth point of $\text{rep}^{{\text{ss}}} R$ then the normal space

$$N(\phi) = T_\phi \text{rep}^{{\text{ss}}} R/T_\phi \mathcal{G}L_n. \phi$$

to the $\mathcal{G}L_n$-orbit decomposes as a $\mu_e$-representation into a direct sum of 1-dimensional simples

$$N(\phi) = \mathbb{C}_{i_1} \oplus \ldots \oplus \mathbb{C}_{i_d}$$

where the action of the generator on $\mathbb{C}_{i_k}$ is by multiplication with $\zeta^{i_k}$. Alternatively, $\phi$ determines a (necessarily smooth) point $[\phi] \in \text{spec} Z(R)$ and because $N(\phi)$ is equal to $\text{Ext}^1_R(S_\phi, S_\phi)$ and because $R$ is Azumaya in $[\phi]$ it coincides with $\text{Ext}^1_Z(S_{[\phi]}, S_{[\phi]})$ (where $S_{[\phi]}$ is the simple 1-dimensional representation of $Z(R)$ determined by $[\phi]$) which is identical to the tangent space $T_{[\phi]} \text{spec} Z(R)$. The action of the stabilizer subgroup $\mu_e$ on $\text{Ext}^1_R(S_\phi, S_\phi)$ carries over to that on $T_{[\phi]} \text{spec} Z(R)$.

The one-parameter family of simple representations also determines a point $\phi \in \text{proj} Z(R)$ and an application of the Luna slice theorem [12] asserts that for all $t \in \mathbb{C}$ there is a neighborhood of $((\phi, t) \in \text{proj} Z(R) \times \mathbb{C}$ which is étale isomorphic to a neighborhood of $0$ in $N(\phi)/\mu_e$, see [7, Thm. 5].

### 3.1. From $\text{Proj}(A)$ to $\text{rep}_A$. In noncommutative projective algebraic geometry, see a.o. [14, 15] and [3], one studies the Grothendieck category $\text{Proj}(A)$ which is the quotient category of all graded left $A$-modules modulo the subcategory of torsion modules. In the case of 3-dimensional Sklyanin algebras the linear modules, that is those with Hilbert series $(1 - t)^{-1}$ (point modules) or $(1 - t)^{-2}$ (line modules) were classified in [5]. Identify $\mathbb{P}^2$ with $\mathbb{P}_{nc}^2 = \mathbb{P}(A_1)$, then

- point modules correspond to points on the elliptic curve $E \subset \mathbb{P}_{nc}^2$
- line modules correspond to lines in $\mathbb{P}_{nc}^2$
In the case of interest to us, when $A$ corresponds to a couple $(E, p)$ with $p$ a torsion point of order $n$ also fat modules are important which are critical cyclic graded left $A$-modules with Hilbert series $n(1-t)^{-1}$. They were classified by M. Artin and are relevant in the study of $\text{proj} Z(A) = \mathbb{P}^2_e = \mathbb{P}(Z(A)_n^*)$. Observe that the reduced norm map $N$ relates the different manifestations of $\mathbb{P}^2$ and the elliptic curve $E$ with its isogenous curve $E/(p)$.

\[
\begin{array}{c}
\mathbb{P}^2_{nc} = \mathbb{P}(A_1^*) & \xrightarrow{N} & \mathbb{P}^2_e = \mathbb{P}(Z(A)_n^*) \\
E & \xrightarrow{\cdot/p} & E' = E/(p)
\end{array}
\]

Points $\rho \in \mathbb{P}^2 - E'$ determine fat points $F_\pi$ with graded endomorphism ring isomorphic to $M_n(\mathbb{C}[t, t^{-1}])$ with $\text{deg}(t) = 1$, and hence determine a one-parameter family of simple $n$-dimensional representations in $\text{rep}^*_n A$ with $\text{GL}_n \times \mathbb{C}^*$-stabilizer subgroup $\mathbb{C}^* \times 1$. There is an effective method to construct $F_\pi$, see [9]. Write $\rho$ as the intersection of two lines $V(z) \cap V(z')$ and let $V(z') \cap E' = \{q_1, q_2, q_3\}$ be the intersection with the elliptic curve $E'$. Then by lifting the $q_i$ through the isogeny to $n$ points $p_{ij} \in E$ we see that we can lift the line $V(z')$ to $n^2$ lines in $\mathbb{P}^2_{nc} = \mathbb{P}(A_1^*)$, that is, there are $n^2$ one-dimensional subspaces $\mathbb{C} l \subset A_1$ with the property that $\mathbb{C} N(l) = \mathbb{C} z'$. The fat point corresponding to $\pi$ is then the shifted quotient of a line module determined by $l$

\[
F_\rho \simeq \frac{A}{A l + A z}[n]
\]

On the other hand, if $q$ is a point on $E'$, then lifting $q$ through the isogeny results in an orbit of $n$ points of $E_1 = \{r, r + p, r + [2]p, \ldots, r + [n-1]p\}$. If $P$ is the point module corresponding to $r \in E$, then the fat point module corresponding to $q$ is

\[
F_q = P \oplus P[1] \oplus P[2] \oplus \ldots \oplus P[n-1]
\]

and the corresponding graded endomorphism ring is isomorphic to $M_n(\mathbb{C}[t, t^{-1}])[0,1,2,\ldots,n-1]$ where $\text{deg}(t) = n$ and hence corresponds to a one-parameter family of simple $n$-dimensional representations in $\text{rep}^*_n A$ with $\text{GL}_n \times \mathbb{C}^*$-stabilizer subgroup generated by $\mathbb{C}^* \times 1$ and a cyclic group of order $n$

\[
\mu_n = \{(1, \zeta, \ldots, \zeta^n)\}
\]

with $\zeta$ a primitive $n$-th root of unity. In fact, we can give a concrete matrix-representation of these simple modules. Assume that $r + [i]p = [a_i : b_i : c_i] \in \mathbb{P}^2_{nc}$
then the fat point module $F_q$ corresponds to the quiver-representation

![Quiver Diagram]

and the map $A \longrightarrow M_n(\mathbb{C}[t,t^{-1}])(0,1,2,\ldots,n-1)$ sends the generators $x, y$ and $z$ to the degree one matrices

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & a_1 & \cdots & 0 \\
0 & 0 & \cdots & a_n \\
0 & 0 & \cdots & 0
\end{pmatrix} \quad \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & b_1 & \cdots & 0 \\
0 & 0 & \cdots & b_n \\
0 & 0 & \cdots & 0
\end{pmatrix} \quad \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & c_1 & \cdots & 0 \\
0 & 0 & \cdots & c_n \\
0 & 0 & \cdots & 0
\end{pmatrix}
$$

**Theorem 4.** Let $A$ be a 3-dimensional Sklyanin algebra corresponding to a couple $(E,p)$ where $p$ is a torsion point of order $n$ and assume that $(n,3) = 1$. Consider the GIT-quotient

$$
\text{rep}_n A \longrightarrow \text{spec}Z(A) = \text{rep}_n A/\mathbb{GL}_n
$$

Then we have

1. $\text{rep}_n^\text{ss} A$ is a smooth variety of dimension $n^2 + 2$
2. $A$ is an Azumaya algebra away from the isolated singularity $\tau \in \text{spec}Z(A)$
3. the nullcone $\pi^{-1}(\tau)$ contains singularities

**Proof.** We know that $\mathbb{P}_n^2 = \text{proj}Z(A) = \text{rep}_n^\text{ss} A/\mathbb{GL}_n \times \mathbb{C}^*$ classifies one-parameter families of semi-stable $n$-dimensional semi-simple representations of $A$. To every point $p \in \mathbb{P}_n^2$ we have associated a one-parameter family of simples, so all semi-stable $A$-representations are in fact simple as the semi-simplification $M^\text{ss}$ of a semi-stable representation still belongs to $\text{rep}_n^\text{ss} A$. But then, all non-trivial semi-simple $A$-representations are simple and therefore the GIT-quotient

$$
\text{rep}_n^\text{ss} A \longrightarrow \text{spec}Z(A) - \{\tau\} = \text{rep}_n^\text{ss} A/\mathbb{GL}_n
$$

is a principal $\mathbb{P}_{\mathbb{GL}}$-fibration in the étale topology. This proves (1).

The second assertion follows as principal $\mathbb{P}_{\mathbb{GL}}$-fibrations in the étale topology correspond to Azumaya algebras. For (3), if $\text{rep}_n A$ would be smooth, the algebra $A$ would be Cayley-smooth as in [10]. There it is shown that the only type of central singularity that can arise for Cayley-smooth algebras with a 3-dimensional center is the conifold singularity.

If we want to distinguish between the two types of simple representations, we have to consider the $\mathbb{GL}_n \times \mathbb{C}^*$-action.

**Lemma 1.** If $S$ is a simple $A$-representation with $\mathbb{GL}_n \times \mathbb{C}^*$-orbit determining a fat point $F_q$ with $q \in E'$, then the normal space $N(S)$ to the $\mathbb{GL}_n$-orbit decomposes...
as representation over the $\text{GL}_n \times \mathbb{C}^*$-stabilizer subgroup $\mu_n$ as $\mathbb{C}_0 \oplus \mathbb{C}_0 \oplus \mathbb{C}_3$, or in the terminology of [7], the associated local weighted quiver is

![Diagram](image)

**Proof.** From [15] we know that the center $Z(A)$ can be represented as

$$Z(A) = \frac{\mathbb{C}[x', y', z', c_3]}{(c_3^3 - \text{cubic}(x', y', z'))}$$

where $x', y', z'$ are of degree $n$ (the reduced norms of $x, y, z$) and $c_3$ is the canonical central element of degree 3. The simple $A$-representation $S$ determines a point $s \in \text{spec}Z(A)$ such that $c_3(s) = 0$. Again, as $A$ is Azumaya over $s$ we have that $N(S) = \text{Ext}^1_A(S, S)$ coincides with the tangent space $T_s\text{spec}Z(A)$. Gradation defines a $\mu_n$-action on $Z(A)$ leaving $x', y', z'$ invariant and sending $c_3$ to $\zeta^3 c_3$. The stabilizer subgroup of this action in $s$ is clearly $\mu_n$ and computing the tangent space gives the required decomposition. \qed

### 3.2. $\mathcal{A}$ is Cayley-smooth.

Because $A$ is a finitely generated module over $Z(A)$, it defines a coherent sheaf of algebras $\mathcal{A}$ over $\text{proj}Z(A) = \mathbb{P}^2$. In this subsection we will show that $\mathcal{A}$ is a sheaf of Cayley-smooth algebras of degree $n$.

As $(n, 3) = 1$ it follows that the graded localisation $Q^p_{x'}(A)$ at the multiplicative set of central elements $\{1, x', x'^2, \ldots\}$ contains central elements $t$ of degree one and hence is isomorphic as a graded algebra to

$$Q^p_{x'}(A) = (Q^p_{x'}(A))_0[t, t^{-1}]$$

By definition $\Gamma(\mathcal{X}(x'), \mathcal{A}) = (Q^p_{x'}(A))_0$ and by the above isomorphism it follows that $\Gamma(\mathcal{X}(x'), \mathcal{A})$ is a Cayley-Hamilton domain of degree $n$ and is Auslander regular of dimension two and consequently a maximal order. Repeating this argument for the other standard opens $\mathcal{X}(y')$ and $\mathcal{X}(z')$ we deduce

**Proposition 1.** $\mathcal{A}$ is a coherent sheaf of Cayley-Hamilton maximal orders of degree $n$ which are Auslander regular domains of dimension 2 over $\text{proj}Z(A) = \mathbb{P}^2_\mathbb{C}$.

Thus, $\mathcal{A}$ is a maximal order over $\mathbb{P}^2$ in a division algebra $\Sigma$ over $\mathbb{C}(\mathbb{P}^2)$ of degree $n$. By the Artin-Mumford exact sequence (see for example [10, 3.6]) describing the Brauer group of $\mathbb{C}(\mathbb{P}^2)$ we know that $\Sigma$ is determined by the ramification locus of $\mathcal{A}$ together with a cyclic $\mathbb{Z}_n$-cover over it.

Again using the above local description of $A$ as a graded algebra over $Z(A)$ we see that the fat point module corresponding to a point $p \notin E'$ determines a simple $n$-dimensional representation of $\mathcal{A}$ and therefore $\mathcal{A}$ is Azumaya in $p$. However, if $p \in E'$, then the corresponding fat point is of the form $P \oplus P[1] \oplus \ldots \oplus P[n-1]$ and this corresponds to a semi-simple $n$-dimensional representation which is the direct sum of $n$ distinct one-dimensional $\mathcal{A}$-representations, one component for each point of $E$ lying over $p$. Hence, we see that the ramification divisor of $\mathcal{A}$ coincides with $E'$ and, naturally, the division algebra $\Sigma$ is the one corresponding to the cyclic $\mathbb{Z}_n$-cover $E \longrightarrow E' = E/\langle \tau \rangle$.

Because $\mathcal{A}$ is a maximal order with smooth ramification locus, we deduce from [10, §5.4]
Proposition 2. $A$ is a sheaf of Cayley-smooth algebras over $\mathbb{P}^2_c$ and hence $\text{rep}_n(A)$ is a smooth variety of dimension $n^2 + 1$ with GIT-quotient

$$\text{rep}_n(A) \xrightarrow{\pi} \mathbb{P}^2_c = \text{rep}_n(A)//\text{GL}_n$$

and is a principal $\text{PGL}_n$-fibration over $\mathbb{P}^2_c - E'$.

3.3. The non-commutative blow-up. Consider the augmentation ideal $\mathfrak{m} = (x, y, z)$ of the 3-dimensional Sklyanin algebra $A$ corresponding to a couple $(E, p)$ with $p$ a torsion point of order $n$. Define the non-commutative blow-up algebra to be the graded algebra

$$B = A \oplus \mathfrak{m}t \oplus \mathfrak{m}^2t^2 \oplus \ldots \subset A[t]$$

with degree zero part $A$ and where the commuting variable $t$ is given degree 1. Note that $B$ is a graded subalgebra of $A[t]$ and therefore is again a Cayley-Hamilton algebra of degree $n$. Moreover, $B$ is a finite module over its center $Z(B)$ which is a graded subalgebra of $Z(A)[t]$. Observe that $B$ is generated by the degree zero elements $x, y, z$ and by the degree one elements $X = xt, Y = yt$ and $Z = zt$. Apart from the Sklyanin relations among $x, y, z$ and among $X, Y, Z$ these generators also satisfy commutation relations such as $Xz = xX, Yz = yY, Xz = zX$ and so on.

With $\text{rep}_{n^*}B$ we will denote again the Zariski open subset of $\text{rep}_nB$ consisting of all trace-preserving $n$-dimensional semi-stable representations, that is, those on which some central homogeneous element of $Z(B)$ of strictly positive degree does not vanish. Theorem 2 asserts that $\text{rep}_{n^*}B$ is a smooth variety of dimension $n^2 + 3$.

Proof of Theorem 2 : As before, we have a $\text{GL}_n \times \mathbb{C}^*$-action on $\text{rep}_{n^*}B$ with corresponding GIT-quotient

$$\text{proj}Z(B) \simeq \text{rep}_{n^*}B//\text{GL}_n \times \mathbb{C}^*$$

Composing the GIT-quotient map with the canonical morphism (taking the degree zero part) $\text{proj}Z(B) \longrightarrow \text{spec}Z(A)$ we have a projection

$$\gamma : \text{rep}_{n^*}B \longrightarrow \text{spec}Z(A)$$

Let $\mathfrak{p}$ be a maximal ideal of $Z(A)$ corresponding to a smooth point, then the graded localization of $B$ at the degree zero multiplicative subset $Z(A) - \mathfrak{p}$ gives

$$B_{\mathfrak{p}} \simeq A_{\mathfrak{p}}[t, t^{-1}]$$

whence $B_{\mathfrak{p}}$ is an Azumaya algebra over $Z(A)[t, t^{-1}]$ and therefore over $\text{spec}Z(A) - \{\tau\}$ the projection $\gamma$ is a principal $\text{PGL}_n \times \mathbb{C}^*$-fibration and in particular the dimension of $\text{rep}_{n^*}B$ is equal to $n^2 + 3$.

This further shows that possible singularities of $\text{rep}_{n^*}B$ must lie in $\gamma^{-1}(\tau)$ and as the singular locus is Zariski closed we only have to prove smoothness in points of closed $\text{GL}_n$-orbits in $\gamma^{-1}(\tau)$. Such a point $\phi$ must be of the form

$$x \mapsto 0, \quad y \mapsto 0, \quad z \mapsto 0, \quad X \mapsto K, \quad Y \mapsto L, \quad Z \mapsto M$$

By semi-stability, $(K, L, M)$ defines a simple $n$-dimensional representation of $A$ and its $\text{GL}_n \times \mathbb{C}^*$-orbit defines the point $[\text{det}(K) : \text{det}(L) : \text{det}(M)] \in \mathbb{P}^2_c$. hence we may assume for instance that $K$ is invertible.

The tangent space $T_\phi \text{rep}_{n^*}B$ is the linear space of all trace-preserving algebra maps $B \longrightarrow M_n(\mathbb{C}[e])$ of the form

$$x \mapsto 0 + eU, y \mapsto 0 + eV, z \mapsto 0 + eW, X \mapsto K + eR, Y \mapsto L + eS, Z \mapsto M + eT$$
and we have to use the relations in \( B \) to show that the dimension of this space is at most \( n^2 + 3 \). As \((K, L, M)\) is a simple \( n \)-dimensional representation of the Sklyanin algebra, we know already that \((R, S, T)\) depend on at most \( n^2 + 2 \) parameters. Further, from the commutation relations in \( B \) we deduce the following equalities (using the assumption that \( K \) is invertible)

- \( xX = Xx \Rightarrow UK = KU \)
- \( xY = Yx \Rightarrow UL = KV \Rightarrow K^{-1}UL = V \)
- \( xZ = Zx \Rightarrow UM = KW \Rightarrow K^{-1}UM = W \)
- \( Yx = yX \Rightarrow LU = VK \Rightarrow LK^{-1}U = V \)
- \( Zx = zX \Rightarrow MU = WK \Rightarrow MK^{-1}U = W \)

These equalities imply that \( K^{-1}U \) commutes with \( K, L \) and \( M \) and as \((K, L, M)\) is a simple representation and hence generate \( M_n(\mathbb{C}) \) it follows that \( K^{-1}U = \lambda I_n \) for some \( \lambda \in \mathbb{C} \). But then it follows that

\[
U = \lambda K, \quad V = \lambda L, \quad W = \lambda M
\]

and so the triple \((U, V, W)\) depends on at most one extra parameter, showing that \( T_0\text{rep}^*_nB \) has dimension at most \( n^2 + 3 \), finishing the proof.

**Remark 1.** The statement of the previous theorem holds in a more general setting, that is, \( \text{rep}^*_nB \) is smooth whenever \( B = A \oplus A^+ t \oplus (A^+)^2 t^2 \oplus \ldots \) with \( A \) a positively graded algebra that is Azumaya away from the maximal ideal \( A^+ \) and \( Z(A) \) smooth away from the origin.

Unfortunately this does not imply that \( \text{proj} Z(B) = \text{rep}^*_nB / \text{GL}_n \times \mathbb{C}^* \) is smooth as there are closed \( \text{GL}_n \times \mathbb{C}^* \) orbits with stabilizer subgroups strictly larger than \( \mathbb{C}^* \times 1 \). This happens precisely in semi-stable representations \( \phi \) determined by

\[
x \mapsto 0, \quad y \mapsto 0, \quad z \mapsto 0, \quad X \mapsto K, \quad Y \mapsto L, \quad Z \mapsto M
\]

with \([\text{det}(K) : \text{det}(L) : \text{det}(M)] \in \mathbb{F}^3\). In which case the matrices \((K, L, M)\) can be brought into the form

\[
\begin{bmatrix}
0 & 0 & \cdots & a_{n-1} & 0 \\
0 & a_0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & a_{n-2} & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}
\]

and the stabilizer subgroup is generated by \( \mathbb{C}^* \times 1 \) together with the cyclic group of order \( n \)

\[
\mu_n = \langle \left( 1, \zeta, \ldots, \right), \zeta \rangle
\]

**Lemma 2.** If \( \phi \) is a representation as above, then the normal space \( N(\phi) \) to the \( \text{GL}_n \)-orbit decomposes as a representation over the \( \text{GL}_n \times \mathbb{C}^* \)-stabilizer subgroup \( \mu_n \) as \( \mathbb{C}_0 \oplus \mathbb{C}_0 \oplus \mathbb{C}_3 \oplus \mathbb{C}_{-1} \), that is, the associated local weighted quiver is

\[
\begin{array}{c|c|c|c}
\text{Type} & \text{Weight} & \text{Quiver} \\
\hline
-1 & 1 & 1 & 1 \\
\end{array}
\]
Proof: The extra tangential coordinate \( \lambda \) determines the tangent-vectors of the three degree zero generators
\[
x \mapsto 0 + \epsilon \lambda K, \quad y \mapsto 0 + \epsilon \lambda L, \quad z \mapsto 0 + \epsilon \lambda M
\]
and so the generator of \( \mu_n \) acts as follows
\[
\begin{bmatrix}
1 \\
\zeta \\
\vdots \\
\zeta^{n-1}
\end{bmatrix}
\cdot (\epsilon \lambda(K, L, M))
= \epsilon \zeta^{n-1} \lambda(K, L, M)
\]
and hence accounts for the extra component \( C_{-1} \).

We have now all information to prove Theorem 3 which asserts that the canonical map
\[
\text{proj}Z(B) \longrightarrow \text{spec}Z(A)
\]
is a partial resolution of singularities, with singular locus \( E' = E/\langle p \rangle \) in the exceptional fiber, all singularities of type \( \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n \). In other words, the isolated singularity of \( \text{spec}Z(A) \) ‘sees’ the elliptic curve \( E' \) and the isogeny \( E \longrightarrow E' \) defining the 3-dimensional Sklyanin algebra \( A \).

**Proof of Theorem 3:** The GIT-quotient map
\[
\text{rep}^*_n B \longrightarrow \text{proj}Z(B)
\]
is a principal \( \text{PGL}_n \times \mathbb{C}^* \)-bundle away from the elliptic curve \( E' \) in the exceptional fiber whence \( \text{proj}Z(B) - E' \) is smooth. The application to the Luna slice theorem of \([7, \text{Thm. 5}]\) asserts that for any point \( \bar{\phi} \in E' \longrightarrow \text{proj}Z(B) \) and all \( t \in \mathbb{C} \) there is a neighborhood of \( (\bar{\phi}, t) \in \text{proj}Z(B) \times \mathbb{C} \) which is \( \acute{e}tale \) isomorphic to a neighborhood of 0 in \( N(\phi)/\mu_n \). From the previous lemma we deduce that
\[
N(\phi)/\mu_n \simeq \mathbb{C} \times \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n
\]
where \( \mathbb{C}[\mathbb{C}^2/\mathbb{Z}_n] \simeq \mathbb{C}[u, v, w]/(w^n - uv^3) \), finishing the proof.

As \( B \) is a finite module over its center, it defines a coherent sheaf of algebras over \( \text{proj}Z(B) \). From Theorem 3 we obtain

**Corollary 1.** The sheaf of Cayley-Hamilton algebras \( B \) on \( \text{proj}Z(B) \) is Azumaya away from the elliptic curve \( E' \) in the exceptional fiber \( \pi^{-1}(m) = \mathbb{P}^2 \) and hence is Cayley-smooth on this open set. However, \( B \) is not Cayley-smooth.

**Proof.** For a point \( p \) in the exceptional fiber \( \pi^{-1}(m) - E' \) we already know that \( \text{proj}Z(B) \) is smooth and that \( B \) is Azumaya, which implies that \( \text{rep}^*_n B \) is smooth in the corresponding orbit. However, for a point \( p \in E' \) we know that \( \text{proj}Z(B) \) has a non-isolated singularity in \( p \). Therefore, \( \text{rep}^*_n B \) can not be smooth in the corresponding orbit, as the only central singularity possible for a Cayley-smooth order over a center of dimension 3 is the conifold singularity, which is isolated. \( \square \)
References


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